Math 274 | IVP Thing | Name: ____________
14 Nov 2017 | Show Appropriate Work | Scaled to $\boxed{10}$ points.

As usual, I hope you finish this in class, if not it is due Wednesday. We are interested in solving the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}, \text{ and } \mathbf{x}_0 = \begin{bmatrix} -8 \\ 6 \\ 22 \end{bmatrix}.$$

1. $\boxed{1}$ Vefify $\mathbf{x}_1 = \begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix}$ solves $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\vec{x}_1{}' = \begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix} \overset{\checkmark}{=} \underline{A}\vec{x}_1 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix} = \begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix}$$

2. $\boxed{1}$ Vefify $\mathbf{x}_2 = \begin{bmatrix} -2e^{2t} \\ e^{2t} \\ 4e^{2t} \end{bmatrix}$ solves $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\vec{x}_2{}' = \begin{bmatrix} -4e^{2t} \\ 2e^{2t} \\ 8e^{2t} \end{bmatrix} \overset{\checkmark}{=} \underline{A}\vec{x}_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -2e^{2t} \\ e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ 2e^{2t} \\ 8e^{2t} \end{bmatrix}$$

3. $\boxed{1}$ Vefify $\mathbf{x}_3 = \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 4e^{3t} \end{bmatrix}$ solves $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\vec{x}_3{}' = \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \\ 12e^{3t} \end{bmatrix} \overset{\checkmark}{=} \underline{A}\vec{x}_3 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -e^{3t} \\ e^{3t} \\ 4e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \\ 12e^{3t} \end{bmatrix}$$

4. $\boxed{1}$ Show $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] \neq 0$.

$$W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = \begin{vmatrix} -e^t & -2e^{2t} & -e^{3t} \\ e^t & e^{2t} & e^{3t} \\ 2e^t & 4e^{2t} & 4e^{3t} \end{vmatrix} = -e^t\left[4e^{5t} - 4e^{5t}\right] + 2e^{2t}\left[4e^{4t} - 2e^{4t}\right] \cancel{+}\, e^{2t}\left[4e^{3t} - 2e^{3t}\right]$$

$$= 4e^{6t} - 2e^{6t} = 2e^{6t} \neq 0$$

5. $\boxed{1}$ Find a fundamental matrix $\mathbf{X}(t)$ for $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ and then evaluate $\mathbf{X}(0)$.

$$\underline{X}(t) = \begin{bmatrix} -e^{t} & -2e^{2t} & -e^{+3t} \\ e^{t} & e^{2t} & e^{3t} \\ 2e^{t} & 4e^{2t} & 4e^{3t} \end{bmatrix} \quad \text{so} \quad \underline{X}(0) = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix}$$

6. $\boxed{3}$ Use row reduction to determine $\mathbf{X}^{-1}(0)$.

$$\left[\begin{array}{ccc|ccc} -1 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \leftarrow (-1)R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 - R_1 \\ \leftarrow R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ \leftarrow (-1)R_2 \\ \leftarrow \frac{R_3}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right] \leftarrow R_1 - 2R_2 - R_3$$

$$\text{so} \quad \underline{X}^{-1}(0) = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix}$$

7. $\boxed{2}$ You will show in the Section 9.4 #28 that $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0)\mathbf{x}_0$ is the solution to the given initial value problem. Using the pieces you have computed, perform this mulitiplication and express $\mathbf{x}(t)$ as a vector function.

$$\underline{X}^{-1}(0)\vec{x}_0 = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -8 \\ 6 \\ 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

so

$$\vec{x}(t) = \underline{X}(t)\underline{X}^{-1}(0)\vec{x}_0 = \begin{bmatrix} -e^{t} & -2e^{2t} & -e^{+3t} \\ e^{t} & e^{2t} & e^{3t} \\ 2e^{t} & 4e^{2t} & 4e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -e^{t} - 4e^{2t} - 3e^{+3t} \\ 2e^{t} + 2e^{2t} + 3e^{3t} \\ 2e^{t} + 8e^{2t} + 12e^{3t} \end{bmatrix}$$