Math 274 Matrix Exponential Thing

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Last time we saw that if $\mathbf{X}(t)$ is a Fundamental Matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ then

$$e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0). \tag{1}$$

For any 2×2 matrix $\mathbf{A}$, the matrix exponential $e^{\mathbf{A}t}$ can be computed according to the table below.

Eigenvalues of $\mathbf{A}$	$e^{\mathbf{A}t}$
r_1, r_2 real and distinct	$e^{r_1 t}\frac{1}{r_1 - r_2}(\mathbf{A} - r_2\mathbf{I}) - e^{r_2 t}\frac{1}{r_1 - r_2}(\mathbf{A} - r_1\mathbf{I})$
r real repeated twice	$e^{rt}\mathbf{I} + te^{rt}(\mathbf{A} - r\mathbf{I})$
$\alpha \pm i\beta$ complex conjugate pair	$e^{\alpha t}\cos(\beta t)\mathbf{I} + e^{\alpha t}\sin(\beta t)\frac{1}{\beta}(\mathbf{A} - \alpha\mathbf{I})$

1. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

   (a) Compute $e^{\mathbf{A}t}$ using (1) above.

$$\begin{vmatrix} 4-r & -2 \\ 1 & 1-r \end{vmatrix} = (4-r)(1-r)+2 = r^2 - 5r + 6 = (r-3)(r-2)$$

$$r_1 = 2: \quad \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\vec{u}_1 = \vec{0}, \text{ choose } \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad r_2 = 3: \quad \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\vec{u}_2 = \vec{0}, \text{ choose } \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} e^{2t} & 2e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix}, \quad \underline{X}(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \underline{X}^{-1}(0) = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{So } e^{At} = \underline{X}\,\underline{X}^{-1}(0) = \begin{bmatrix} e^{2t} & 2e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (2e^{3t} - e^{2t}) & (2e^{2t} - 2e^{3t}) \\ (e^{3t} - e^{2t}) & (2e^{2t} - e^{3t}) \end{bmatrix}$$

   (b) Compute $e^{\mathbf{A}t}$ using the formuala.

Let $r_1 = 3$, $r_2 = 2$ so $\frac{1}{r_1 - r_2} = 1$

$$e^{At} = e^{3t}\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) - e^{2t}\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (2e^{3t} - e^{2t}) & (2e^{2t} - 2e^{3t}) \\ (e^{3t} - e^{2t}) & (2e^{2t} - e^{3t}) \end{bmatrix}$$

   (c) Note, $e^{\mathbf{A}t}$ is unique, so your solutions should be the same.

2. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -6 & -1 \end{bmatrix}$.

(a) Compute $e^{\mathbf{A}t}$ using (1).

$$\begin{vmatrix} 5-r & 3 \\ -6 & -1-r \end{vmatrix} = (5-r)(-1-r) + 18 = r^2 - 4 + 13 = (r-2)^2 + 3^2 = 0 \quad \text{so} \quad r = 2 \pm 3i$$

$$\begin{bmatrix} 3-3i & 3 \\ -6 & -3-3i \end{bmatrix} \vec{z} = \vec{0} \quad \text{choose } \vec{z} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} e^{2t}\cos 3t & e^{2t}\sin 3t \\ e^{2t}(-\cos 3t - \sin 3t) & e^{2t}(\cos 3t - \sin 3t) \end{bmatrix} \quad \text{so} \quad \underline{X}(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \underline{X}^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = \underline{X}\,\underline{X}^{-1}(0) = \begin{bmatrix} e^{2t}(\cos 3t + \sin 3t) & e^{2t}\sin 3t \\ -2e^{2t}\sin 3t & e^{2t}(\cos 3t - \sin 3t) \end{bmatrix}$$

(b) Compute $e^{\mathbf{A}t}$ using the formuala.

$$e^{At} = e^{2t}\cos 3t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{3} e^{2t} \sin 3t \begin{bmatrix} 3 & 3 \\ -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}(\cos 3t + \sin 3t) & e^{2t}\sin 3t \\ -2e^{2t}\sin 3t & e^{2t}(\cos 3t - \sin 3t) \end{bmatrix}$$

(c) Solve the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Since e^{At} is a fundamental matrix the general solution is $\vec{x} = e^{At}\vec{c}$. To solve for $\vec{c}$ we substitute $t=0$. The nice property of e^{At} is that $e^{A0} = I$, so $\vec{x}(0) = I\vec{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ so $\vec{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The solution to the initial value problem is then $\vec{x} = e^{At}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

3. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$.

(a) In Section 9.5 #35 we showed that $\mathbf{A}$ had a repeated eigenvalue and found a solution making use of the idea of a *generalized eigenvector*. This was a non-trivial exercise.

(b) Compute $e^{\mathbf{A}t}$ using the formuala.

$$\begin{vmatrix} 1-r & -1 \\ 4 & -3-r \end{vmatrix} = (1-r)(-3-r) + 4 = r^2 + 2r + 1 = (r+1)^2 \quad \text{so} \quad r = -1 \text{ (repeated)}$$

$$e^{At} = e^{t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{t}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} e^t + 2te^t & -te^t \\ 4te^t & e^t - 2te^t \end{bmatrix}$$

$$\text{-- OR --} \quad e^t \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix}$$