Math 274 Section:_____ Name:__________________

Midterm Exam - Tuesday, 10 October 2017

**Show all appropriate work to receive full credit.** Point values are in $\boxed{\text{boxes}}$

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at Montana State University, and that I have neither given nor received any unauthorized assistance (calculator, cell phone, notes, other people's exams, etc.) on this exam. Furthermore, I have and will not discuss the content of this exam, or provide any written or verbal aid to others. I understand that if I violate this statement, I will receive an exam score of 0."

SIGNATURE:______________________________

1. Find a general solution for the following equations.

   (a) $\boxed{5}$ $x'' - 6x' + 9x = 0$

   $r^2 - 6r + 9 = 0$

   $(r-3)^2 = 0$

   $x = C_1 e^{3t} + C_2 t e^{3t}$

   (b) $\boxed{5}$ $y'' - 6y' + 13y = 0$

   $r^2 - 6r + 9 = -4$

   $y = C_1 e^{3t} \cos 2t + C_2 e^{3t} \sin 2t$

   $r = 3 \pm 2i$

   (c) $\boxed{5}$ $t^2 x'' - 2x = 0$

   $x = C_1 t^2 + C_2 t^{-1}$

   $r^2 - r - 2 = 0$

   $(r-2)(r+1) = 0$

2. For each of the following, specify the form of a particular solution suggested by the Method of Undetermined Coefficients. **DO NOT SOLVE FOR THE CONSTANTS.**

   (a) $\boxed{4}$ $y'' = (2t+3)e^{3t} + 7$

   $y_p = (At+B)e^{3t} + Ct^2$

   $r^2 = 0$

   (b) $\boxed{6}$ $x'' - 9x = te^{3t} \cos t$

   $y_p = (At+B)e^{3t} \cos t + (Ct+D)e^{3t} \sin t$

   $r^2 - 9 = 0$

3. Consider a Mass-Spring system governed by the equation

$$y'' + by' + 4y = 0,$$

i.e. the mass is 1 and the spring constant is 4 in appropriate units, and there are no external forces.

(a) $\boxed{2}$ For what values of $b \geq 0$, the friction constant, does the system have solutions that do not oscillate?

$$b^2 - 4 \cdot 1 \cdot 4 \geq 0$$

so $b \geq 4$

(b) $\boxed{8}$ If there is no friction in the system, i.e. $b = 0$, find the solution satisfying the initial conditions $y(0) = y'(0) = 1$.

$$y'' + 4y = 0$$

$$r^2 + 4 = 0$$

$$y = C_1 \cos 2t + C_2 \sin 2t$$

$y(0) = 1$ so $C_1 = 1$

$y'(0) = 1$ so $C_2 = \frac{1}{2}$

$$y = \cos 2t + \frac{1}{2} \sin 2t$$

4. $\boxed{15}$ Consider a Mass-Spring system governed by the equation

$$y'' + 4y = 8\cos 2t,$$

find a particular solution.

$$y_p = At\cos 2t + Bt\sin 2t$$

$$y_p' = A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t$$

$$= (A + 2Bt)\cos 2t + (B - 2At)\sin 2t$$

$$y_p'' = 2B\cos 2t - (2A + 4Bt)\sin 2t$$

$$+ (2B - 4At)\cos 2t - 2A\sin 2t$$

$$= (4B - 4At)\cos 2t + [-4A - 4Bt]\sin 2t$$

substituting yields

$$y_p'' + 4y_p = [4B - 4At]\cos 2t + [-4A - 4Bt]\sin 2t + 4At\cos 2t + 4Bt\sin 2t$$

$$= 4B\cos 2t - 4A\sin 2t$$

so $B = 2$, $A = 0$ $\Rightarrow$ $y_p = 2t\sin 2t$

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$$z_p = Ate^{2it}$$

$$z_p' = (A + 2iAt)e^{2it}$$

$$z_p'' = [2Ai + 2Ai - 4At]e^{2it}$$

$$= [4Ai - 4At]e^{2it}$$

Substituting yields

$$z_p'' + 4z_p = [4Ai - 4At]e^{2it} + 4Ate^{2it}$$

$$= 4Aie^{2it}$$

so $A = \frac{8}{4i} = -2i$

$$z_p = -2ite^{2it}$$

$$y_p = \mathrm{Re}(z_p) = 2t\sin 2t$$

5. $\boxed{20}$ Given that $y = C_1 t^{-2} + C_2$ is a general solution to

$$t^2 y'' + 3ty' = 0,$$

find a particular solution to

$$t^2 y'' + 3ty' = 4t^{-2} \ln t.$$

$y_1 = t^{-2}$

$y_2 = 1$

$$W = \begin{vmatrix} t^{-2} & 1 \\ -2t^{-3} & 0 \end{vmatrix} = 2t^{-3}$$

In standard form $g = 4t^{-4} \ln t.$

$$V_1 = \int \frac{(-4t^{-4} \ln t)}{2t^{-3}}\, dt = -2 \int \frac{\ln t}{t}\, dt = -(\ln t)^2$$

$$V_2 = \int \frac{(4t^{-4} \ln t)\, t^{-2}}{2t^{-3}}\, dt = \int 2t^{-3} \ln t\, dt = -t^{-2} \ln t - \frac{t^{-2}}{2}$$

$u = \ln t \qquad dv = 2t^{-3}$

$du = \frac{1}{t} dt \qquad v = -t^{-2}$

so $y_p = -t^{-2}(\ln t)^2 - t^{-2} \ln t - \frac{t^{-2}}{2}$ ← Note: the last term is part of the homogeneous solution, so not needed.

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If they fail to put in std form they should get

$$V_1 = \int \frac{(-4t^{-2} \ln t)}{2t^{-3}}\, dt = \int -2t \ln t\, dt = -t^2 \ln t + \frac{t^2}{2}$$

$u = \ln t \qquad dv = -2t\, dt$

$du = \frac{1}{t} dt \qquad v = -t^2$

$$V_2 = \int \frac{(4t^{-2} \ln t)\, t^{-2}}{2t^{-3}}\, dt = \int \frac{2 \ln t}{t}\, dt = (\ln t)^2$$

so $y_p = (\ln t)^2 - \ln t + \frac{1}{2}$

6. $\boxed{10}$ Solve the initial value problem.

$$\frac{dx}{dt} = 1 - e^{-t/20} - \frac{x}{20}, \qquad x(0) = 30$$

$$\frac{dx}{dt} + \frac{1}{20}x = 1 - e^{-t/20} \qquad \mu(t) = e^{t/20}$$

$$x = e^{-t/20} \int (e^{t/20} - 1)\, dt$$

$$= e^{-t/20} \left[ 20e^{t/20} - t + C \right]$$

$$= 20 + (C - t)e^{-t/20}$$

$x(0) = 30$, so $C = 10$

$$x = 20 + (10 - t)e^{-t/20}$$

7. A pot of tapioca pudding is taken off the stove at a temperature of $T_0$ and set to cool in a room of temperature $R$. The temperature of pudding can be modeled by the equation

$$T' = k(R - T) \tag{1}$$

where $k > 0$ is a constant.

Verify that

$$T(t) = R - (R - T_0)e^{-kt} \tag{2}$$

solves equation (1) with initial condition $T(0) = T_0$. In other words,

(a) $\boxed{2}$ verify (2) satisfies $T(0) = T_0$, and

$$T(0) = R - (R - T_0) = T_0 \checkmark$$

(b) $\boxed{6}$ verify that (2) satisfies (1).

$$T' = -(R - T_0)e^{-kt}(-k) = k(R - T_0)e^{-kt} \checkmark$$

$$k(R - T) = k\left(R - R + (R - T_0)e^{-kt}\right) = k(R - T_0)e^{-kt}$$

8. $\boxed{12}$ **Use an appropriate substitution** to solve the initial value problem.

$$\frac{dU}{dt} = U(1 - U^5), \qquad U(0) = \frac{1}{2}.$$

$$\frac{dU}{dt} = U - U^6$$

$$\frac{dU}{dt} - U = -U^6$$

$$U^{-6}\frac{dU}{dt} - U^{-5} = -1$$

$$-5U^{-6}\frac{dU}{dt} + 5U^{-5} = 5$$

Let $v = U^{-5}$

$$\frac{dv}{dt} = -5U^{-6}\frac{dU}{dt}$$

$$\frac{dv}{dt} + 5v = 5 \longrightarrow$$

$v' + 5v = 0$ has solution $v = Ce^{-5t}$

$v' + 5v = 5$ has solution $v = Ce^{-5t} + 1$

by MUCs / inspection

$\downarrow$ or

$\mu(t) = 5t$ so

$$v = e^{-5t}\int 5e^{5t}\,dt$$

$$= e^{-5t}\left(e^{5t} + c\right)$$

$$= 1 + Ce^{-5t}$$

so $U = (1 + Ce^{-5t})^{-1/5}$

$U(0) = \frac{1}{2}$ so

$$\frac{1}{2} = (1 + C)^{-1/5}$$

or $C = \left(\frac{1}{2}\right)^{-5} - 1 = 31$

so $U = \left(1 + 31e^{-5t}\right)^{-1/5}$

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Value | 15 | 10 | 10 | 15 | 20 | 10 | 8 | 12 | 100 |
| Points | | | | | | | | | |