Math 274 | Systems of MUCs | Name: ____________
Due: 27 Nov 2017 | Show Appropriate Work | Point Values in $\boxed{\text{boxes}}$.

As usual, I hope you finish this in class, if not it is due Monday. We are interested in solving the nonhomogeneous equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(\mathbf{t})$ where

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}, \text{ and } \mathbf{f}(t) = \begin{bmatrix} 25e^{3t} \\ 30 \end{bmatrix}.$$

1. $\boxed{2}$ Find a general solution to the homogeneous equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\begin{vmatrix} 2-r & 2 \\ 2 & -1-r \end{vmatrix} = (2-r)(-1-r) - 4 = r^2 - r - 6 = (r-3)(r+2) = 0$$

$$r_1 = 3: \quad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{0}$$

$$\vec{x} = c_1 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$r_2 = -2: \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{0}$$

2. $\boxed{2}$ Use the Method of Undetermined Coefficients to find a particular solution to

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 30 \end{bmatrix}.$$

Try $\vec{x}_q = \begin{bmatrix} a \\ b \end{bmatrix}$ so $\vec{x}_q' = \vec{0}$. Substituting yields

$$\vec{0} = \mathbb{A}\,\vec{x}_q + \begin{bmatrix} 0 \\ 30 \end{bmatrix}$$

so

$$\vec{x}_q = \mathbb{A}^{-1}\begin{bmatrix} 0 \\ -30 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -30 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -60 \\ 60 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \end{bmatrix}$$

3. We are interested in using the Method of Undetermined Coefficients to find a particular solution to

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix}.$$

(a) $\boxed{2}$ We would normally try a function of the form $\mathbf{x}_p(t) = e^{3t}\mathbf{a}$, but in this case the homogeneous equation has a solution of that form. Following what we did in chapter 4, we try $\mathbf{x}_p(t) = te^{3t}\mathbf{a}$. Show this does not work.

$\vec{x}_p = te^{3t}\begin{bmatrix} a \\ b \end{bmatrix}$ so $\vec{x}_p{}' = e^{3t}\begin{bmatrix} a \\ b \end{bmatrix} + 3te^{3t}\begin{bmatrix} a \\ b \end{bmatrix}$. Substituting yields

$e^{3t}\begin{bmatrix} a(1+3t) \\ b(1+3t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} te^{3t}\begin{bmatrix} a \\ b \end{bmatrix} + e^{3t}\begin{bmatrix} 25 \\ 0 \end{bmatrix}$ which is four equations.

top row te^{3t}: $3a = 2a + 2b$ bottom row te^{3t}: $3b = 2a - b$

top row e^{3t}: $a = 25$ bottom row e^{3t}: $b = 0$,

So $a = 25$, $b = 0$, but $a = 2b$, clearly false!

(b) $\boxed{3}$ The proper form for this system is $\mathbf{x}_p(t) = te^{3t}\mathbf{a} + e^{3t}\mathbf{b}$. Use this to find a particular solution.

$\vec{x}_p = te^{3t}\begin{bmatrix} a \\ b \end{bmatrix} + e^{3t}\begin{bmatrix} c \\ d \end{bmatrix}$ so $\vec{x}_p{}' = e^{3t}\begin{bmatrix} a(1+3t) + 3c \\ b(1+3t) + 3d \end{bmatrix}$. Substituting yields

$e^{3t}\begin{bmatrix} 3at + (a+3c) \\ 3bt + (b+3d) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} e^{3t}\begin{bmatrix} at + c \\ bt + d \end{bmatrix} + e^{3t}\begin{bmatrix} 25 \\ 0 \end{bmatrix}$

(1) top te^{3t} : $3a = 2a + 2b \Rightarrow a = 2b$

(2) top e^{3t} : $a + 3c = 2c + 2d + 25$

(3) bottom te^{3t} : $3b = 2a - b \Rightarrow a = 2b$

(4) bottom e^{3t} : $b + 3d = 2c - d$

Substituting (1) into (2) & multiplying by 2 gives

$4b = -2c + 4d + 50$

adding (4) gives

$b = 2c - 4d$

$5b = 50 \Rightarrow b = 10$

so $a = 20$

and $5 = c - 2d$

choose $c = 5$, $d = 0$

[or $c = 1$, $d = -2$ or ...]

4. $\boxed{1}$ Find a general solution to the system

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}, \text{ and } \mathbf{f}(t) = \begin{bmatrix} 25e^{3t} \\ 30 \end{bmatrix}.$$

$\vec{x} = C_1 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -10 \\ 10 \end{bmatrix} + te^{3t}\begin{bmatrix} 20 \\ 10 \end{bmatrix} + e^{3t}\begin{bmatrix} 5 \\ 0 \end{bmatrix}$