Math 274 Complex MUCs, An Example
Section: 4.4
Our goal is to find a particular solution to

$$y'' - y' - 6y = 50t\cos t. \tag{1}$$

Real Method of Undermined Coefficients.
We can find a solution using the Method of Undermined Coefficients. Starting with the appropriate form and differentiating gives the following.

$$\begin{aligned}
y_p &= (At+B)\cos t + (Ct+D)\sin t \\
y_p' &= A\cos t - (At+B)\sin t + C\sin t + (Ct+D)\cos t \\
&= (Ct+D+A)\cos t + (C-At-B)\sin t \\
y_p'' &= C\cos t - (Ct+D+A)\sin t - A\sin t + (C-At-B)\cos t \\
&= (2C-At-B)\cos t + (-Ct-D-2A)\sin t
\end{aligned}$$

Substituting into (1) gives

$$\begin{aligned}
y_p'' - y_p' - 6y_p = {} & (2C-At-B)\cos t + (-Ct-D-2A)\sin t \\
& - (Ct+D+A)\cos t - (C-At-B)\sin t - 6(At+B)\cos t - 6(Ct+D)\sin t.
\end{aligned}$$

Equating coefficients gives the following system of equations.

$$\begin{aligned}
t\cos t: &\qquad -7A - C = 50 &(2)\\
t\sin t: &\qquad A - 7C = 0 &(3)\\
\cos t: &\qquad 2C - A - 7B - D = 0 &(4)\\
\sin t: &\qquad B - 2A - C - 7D = 0 &(5)
\end{aligned}$$

Multiplying equation (3) by 7 and adding it to (2) gives $-50C = 50$, so $C = -1$ and $A = -7$. Substituting those values into (4) and (5) and simplifying gives

$$7B + D = 5 \text{ and } B - 7D = -15.$$

A similar multiplication by 7 and addition leads to $B = 2/5$ and $D = 11/5$. A particular solution is then given by

$$y_p = \left(-7t + \frac{2}{5}\right)\cos t + \left(-t + \frac{11}{5}\right)\sin t.$$

Complex Method of Undermined Coefficients.
In this case, and in general when the inhomogeneity involves a $\cos(\beta t)$ or a $\sin(\beta t)$, it can be convenient to consider the complex version of the problem. Since $\mathrm{Re}(50te^{it}) = 50t\cos t$ we consider the related complex valued equation

$$z'' - z' - 6z = 50te^{it} \tag{6}$$

and try to find a complex valued solution using the same basic method. Once we find a complex solution z_p to (6), we argue that $\mathrm{Re}(z_p)$ solves (1).
Proceeding as before, we start with the appropriate form and differentiate to find the following.

$$\begin{aligned}
z_p &= (At+B)e^{it} \\
z_p' &= [A + (At+B)i]e^{it} \\
z_p'' &= [2Ai - (At+B)]e^{it}
\end{aligned}$$

It is worth noting that there seem to be half as many constants in this case. While true, the constants are now complex valued, i.e. $A = a_1 + ia_2$, so that each has two components.

Substituting into (6) gives

$$z_p'' - z_p' - 6z_p = [2Ai - (At + B)]e^{it} - [A + (At + B)i]e^{it} - 6(At + B)e^{it}.$$

Equating coefficients gives the following system of equations.

$$te^{it}: \qquad A(-7 - i) = 50 \tag{7}$$

$$e^{it}: \qquad A(2i - 1) - B(7 + i) = 0 \tag{8}$$

From (7) we immediately see that $A = 50/(-7-i)$. Recall that division of complex numbers is computed by multiplying by the conjugate. It is then routine to show $A = (-7 + i)$. [Try it yourself.]
From (8) it is equally clear that $B = A(2i - 1)/(7 + i)$. Substituting the value of A and performing the necessary arithmetic gives $B = 2/5 - 11i/5$. [This is where all the work is, do it!]
A particular solution to (6) is then given by

$$z_p = \left[(-7 + i)t + \left(\frac{2}{5} - \frac{11i}{5}\right)\right] e^{it}.$$

To find a particular solution to (1) we take the real part of z_p and find

$$\begin{aligned} y_p &= \mathrm{Re}(z_p) \\ &= \mathrm{Re}\left(\left[(-7 + i)t + \left(\frac{2}{5} - \frac{11i}{5}\right)\right] e^{it}\right) \\ &= \mathrm{Re}\left(\left[(-7 + i)t + \left(\frac{2}{5} - \frac{11i}{5}\right)\right] (\cos t + i \sin t)\right) \\ &= \left(-7t + \frac{2}{5}\right) \cos t + \left(-t + \frac{11}{5}\right) \sin t. \end{aligned}$$