Math 274 Complex MUCs, An Example
Section: 4.4

Our goal is to find a particular solution to

\[ y'' - y' - 6y = 50t \cos t. \]  \hspace{1cm} (1)

**Real Method of Undermined Coefficients.**

We can find a solution using the Method of Undermined Coefficients. Starting with the appropriate form and differentiating gives the following.

\[ y_p = (At + B) \cos t + (Ct + D) \sin t \]
\[ y'_p = A \cos t - (At + B) \sin t + C \sin t + (Ct + D) \cos t \]
\[ y''_p = C \cos t - (Ct + D + A) \sin t - A \sin t + (C - At - B) \cos t \]

Substituting into (1) gives

\[ y''_p - y'_p - 6y_p = (2C - At - B) \cos t + (-Ct - D - 2A) \sin t \]

- \[ (Ct + D + A) \cos t - (C - At - B) \sin t - 6(At + B) \cos t - 6(Ct + D) \sin t. \]

Equating coefficients gives the following system of equations.

\[ t \cos t : \hspace{1cm} -7A - C = 50 \] \hspace{1cm} (2)
\[ t \sin t : \hspace{1cm} A - 7C = 0 \] \hspace{1cm} (3)
\[ \cos t : \hspace{1cm} 2C - A - 7B - D = 0 \] \hspace{1cm} (4)
\[ \sin t : \hspace{1cm} B - 2A - C - 7D = 0 \] \hspace{1cm} (5)

Multiplying equation (3) by 7 and adding it to (2) gives \(-50C = 50\), so \(C = -1\) and \(A = -7\). Substituting those values into (4) and (5) and simplifying gives

\(7B + D = 5\) and \(B - 7D = -15\).

A similar multiplication by 7 and addition leads to \(B = 2/5\) and \(D = 11/5\). A particular solution is then given by

\[ y_p = \left(-7t + \frac{2}{5}\right) \cos t + \left(-t + \frac{11}{5}\right) \sin t. \]

**Complex Method of Undermined Coefficients.**

In this case, and in general when the inhomogeneity involves a \(\cos(\beta t)\) or a \(\sin(\beta t)\), it can be convenient to consider the complex version of the problem. Since Re\((50te^{it}) = 50t \cos t\) we consider the related complex valued equation

\[ z'' - z' - 6z = 50te^{it} \] \hspace{1cm} (6)

and try to find a complex valued solution using the same basic method. Once we find a complex solution \(z_p\) to (6), we argue that Re\((z_p)\) solves (1).

Proceeding as before, we start with the appropriate form and differentiate to find the following.

\[ z_p = (At + B)e^{it} \]
\[ z'_p = [A + (At + B)i]e^{it} \]
\[ z''_p = [2Ai - (At + B)]e^{it} \]

It is worth noting that there seem to be half as many constants in this case. While true, the constants are now complex valued, i.e. \(A = a_1 + ia_2\), so that each has two components.
Substituting into (6) gives

\[ z_p'' - z_p' - 6z_p = [2Ai - (At + B)]e^{it} - [A + (At + B)i]e^{it} - 6(At + B)e^{it}. \]

Equating coefficients gives the following system of equations.

\[
\begin{align*}
t e^{it} & : A(-7 - i) = 50 \quad (7) \\
e^{it} & : A(2i - 1) - B(7 + i) = 0 \quad (8)
\end{align*}
\]

From (7) we immediately see that \( A = 50/(-7 - i). \) Recall that division of complex numbers is computed by multiplying by the conjugate. It is then routine to show \( A = (-7 + i). \) [Try it yourself.]

From (8) it is equally clear that \( B = A(2i - 1)/(7 + i). \) Substituting the value of \( A \) and performing the necessary arithmetic gives \( B = 2/5 - 11i/5. \) [This is where all the work is, do it!]

A particular solution to (6) is then given by

\[ z_p = \left[ (-7 + i)t + \left( \frac{2}{5} - \frac{11i}{5} \right) \right] e^{it}. \]

To find a particular solution to (1) we take the real part of \( z_p \) and find

\[
y_p = \text{Re}(z_p) \\
= \text{Re} \left( \left[ (-7 + i)t + \left( \frac{2}{5} - \frac{11i}{5} \right) \right] e^{it} \right) \\
= \text{Re} \left( \left[ (-7 + i)t + \left( \frac{2}{5} - \frac{11i}{5} \right) \right] \cos t + i \sin t \right) \\
= \left( -7t + \frac{2}{5} \right) \cos t + \left( -t + \frac{11}{5} \right) \sin t.
\]