Math 274 **Section:** ______ **Name:** ____________________

Due: 27 Nov 2018 Point values in $\boxed{\text{boxes}}$.

1. For $t > 0$, consider the following

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -1/t & (t+1)/t \end{bmatrix}, \mathbf{x}_1(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \text{ and } \mathbf{x}_2(t) = \begin{bmatrix} t+1 \\ 1 \end{bmatrix}.$$

(a) $\boxed{3}$ Show $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a fundamental solution set[1] for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

$$\vec{x}_1{}' = \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \quad \underline{A}\vec{x}_1 = \begin{bmatrix} 0 & 1 \\ -1/t & 1+\frac{1}{t} \end{bmatrix}\begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} e^t \\ -\frac{e^t}{t} + e^t + \frac{e^t}{t} \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

so $\vec{x}_1{}' = \underline{A}\vec{x}_1$

$$\vec{x}_2{}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{A}\vec{x}_2 = \begin{bmatrix} 0 & 1 \\ -1/t & 1+1/t \end{bmatrix}\begin{bmatrix} t+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1-\frac{1}{t}+1+\frac{1}{t} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so $\vec{x}_2{}' = \underline{A}\vec{x}_2$

$$W[\vec{x}_1, \vec{x}_2] = \det\begin{bmatrix} e^t & t+1 \\ e^t & 1 \end{bmatrix} = e^t - te^t - e^t = -te^t \neq 0 \quad \text{for } t>0.$$

(b) $\boxed{2}$ Find the solution to the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \mathbf{x}(1) = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

$$\vec{x} = c_1\begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} t+1 \\ 1 \end{bmatrix} \quad \text{at } t=1 \text{ we have } \begin{bmatrix} 7 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} e \\ e \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $7 = ec_1 + 2c_2$

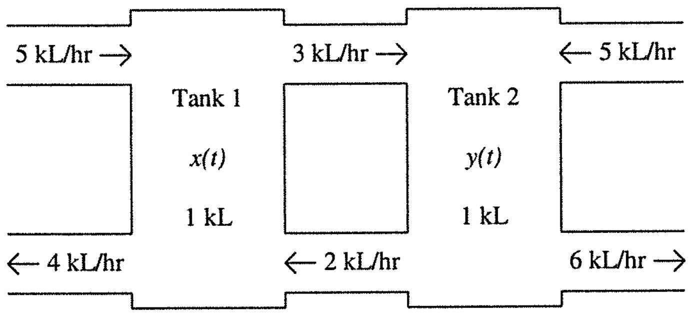
$4 = ec_1 + c_2$

$3 = c_2$ so $c_1 = e^{-1}$

$$\vec{x} = \begin{bmatrix} e^{t-1} \\ e^{t-1} \end{bmatrix} + 3\begin{bmatrix} t+1 \\ 1 \end{bmatrix}$$

[1] Show: (i) $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, and (ii) they are linearly independent. (Use the Wronskian.)

2. Two tanks are initially filled with 1 kL of pure water. A solution with 10 kg/kL of salt is flowing into tank 1 at 5 kL/hr. A solution with 20 kg/kL of salt is flowing into tank 2 at 5 kL/hr. Both tanks are well mixed. The resulting solution is flowing from tank 1 into tank 2 at 3 kL/hr, and from tank 2 into tank 1 at 2 kL/hr. Tank 1 is being drained at 4 kL/hr and tank 2 is being drained at 6 kL/hr. Let $x(t)$ be the amount of salt in tank 1 in kg, and $y(t)$ be the amount of salt in tank 2 in kg.



(a) $\boxed{2}$ Set up an initial value problem that models the amount of salt in each tank.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ 3 & -8 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$\boxed{1}$ Identify the x-nullcline(s), the y-nullcline(s), and any equilibrium[2].

x-null:

$-7x + 2y + 50 = 0$

$y = \frac{7}{2}x - 25$

y-null:

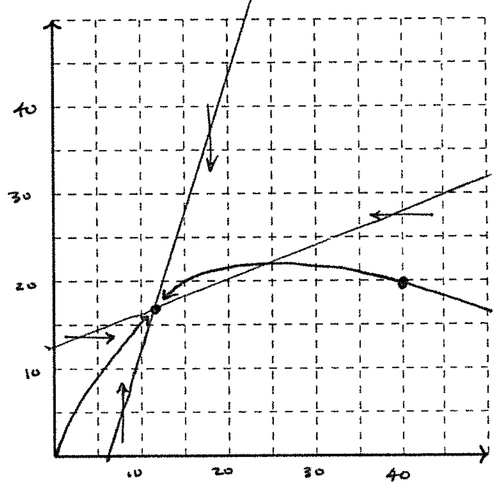
$3x - 8y + 100 = 0$

$y = \frac{3}{8}x + \frac{25}{2}$

equilibrium

$(12, 17)$

$\boxed{1}$ Carefully sketch the phase plane for this system for $[0, 50] \times [0, 50]$. Include the null-clines (with direction arrows) and equilibrium you found above. Also include the solution curves that satisfy the initial data $[0, 0]^T$ and $[40, 20]^T$.



$\boxed{1}$ In a sentence or two, explain what the equilibrium solution means in this system.

The equilibrium, $x = 12$, $y = 17$, gives us the amount of salt in each tank as $t \to \infty$.

[2] Your equilibrium solution should have integer values for each component.