Math 274 Exam 1 **Name:** ______________________

19 Sept 2018 Point values in $\boxed{\text{boxes}}$.

1. For each of the following first order equations determine if it is Separable, Linear, and/or Neither. Check each box that applies. (More than one may apply.)

(a) $\boxed{3}$ $\frac{dx}{dt} = \frac{t-x}{t+1}$

$= \frac{t}{t+1} - \frac{x}{t+1}$

Separable	Linear	Neither
	X	

(b) $\boxed{3}$ $y' = by$, for $b > 0$

Separable	Linear	Neither
X	X	

(c) $\boxed{3}$ $\frac{dy}{dx} = y + \frac{e^y}{x}$

Separable	Linear	Neither
		X

(d) $\boxed{3}$ $y^2 \frac{dy}{dt} = t^2 + 3$

Separable	Linear	Neither
X		

2. Consider the equation

$$\frac{dx}{dt} = 1 - \frac{x}{5}. \tag{1}$$

(a) $\boxed{6}$ Equation (1) is separable, find an explicit general solution using that method.

$$\frac{dy}{dt} = \frac{5-x}{5}$$

$$\int \frac{dx}{5-x} = \int \frac{dt}{5}$$

$$-\ln|5-x| = \frac{t}{5} + C$$

$$5 - x = Ce^{-t/5}$$

$$x = 5 + Ce^{-t/5}$$

(b) $\boxed{6}$ Equation (1) is first order linear, find an explicit general solution using an integrating factor or variation of parameters.

$$\frac{dx}{dt} - \left(-\frac{x}{5}\right) = 1 \qquad u(t) = e^{t/5}$$

$$\left(e^{t/5} x\right)' = e^{t/5}$$

$$x = e^{-t/5}\left(5e^{t/5} + C\right)$$

$$= 5 + Ce^{-t/5}$$

3. Equation (1) can also be written

$$(x-5)dt + 5dx = 0. \tag{2}$$

(a) $\boxed{4}$ Show equation (2) is not exact.

$\frac{\partial}{\partial x}(x-5) = 1 \neq \frac{\partial}{\partial t}(5) = 0$

(b) $\boxed{6}$ Multiplying (2) by the integrating factor $\mu = e^{t/5}$ results in the equation

$$e^{t/5}(x-5)dt + 5e^{t/5}dx = 0$$

which is exact. Find a general solution to the resulting equation using that method.

$F(x,t) = 5e^{t/5}(x-5)$

so

$5e^{t/5}(x-5) = C$

is a general solution.

4. Consider the initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0.$$

(a) $\boxed{6}$ Show that $f(t) = 0$ and $g(t) = t^3$ are both solutions to the initial value problem.

$f'(t) = 0 = 3(0)^{2/3}$ ✓ $\quad f(0) = 0$ ✓

$g'(t) = 3t^2 = 3(t^3)^{2/3}$ ✓ $\quad g(0) = 0$ ✓

(b) $\boxed{4}$ Explain the above in reference to the Uniqueness Theorem.

There is not a unique solution at $y(0) = 0$ since

$\frac{\partial}{\partial y}(3y^{2/3}) = 2y^{-1/3}$ is not continuous for $y = 0$

5. $\boxed{14}$ For parameter $p > 0$, show that $y(t) = p/(1+e^{-pt})$ is a solution to the initial value problem

$$y' = y(p-y), \quad y(0) = p/2.$$

$$y(0) = \frac{p}{1+1} = \frac{p}{2}$$

$$(*) \quad y' = \frac{-p}{(1+e^{-pt})^2}(-pe^{-pt}) = \frac{p^2 e^{-pt}}{(1+e^{-pt})^2}$$

$$(**) \quad y(p-y) = \frac{p}{(1+e^{-pt})}\left(p - \frac{p}{1+e^{-pt}}\right)$$

$$= \frac{p\left(p(1+e^{-pt}) - p\right)}{(1+e^{-pt})^2}$$

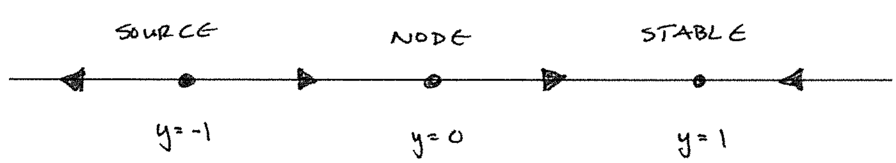
$$= \frac{p^2 e^{-pt}}{(1+e^{-pt})^2}$$

Since $(*) = (**)$ $y(t)$ is a solution.

6. Consider the autonomous equation

$$y' = y^2 - y^4. = y^2(1-y^2)$$

(a) $\boxed{7}$ For the equation, label the equilibria and classify each as a sink (stable), source (unstable), or node (neither) on the phase line. Include direction arrows.



(b) $\boxed{3}$ Predict the asymptotic behavior as $t \to \infty$ of the solution satisfying $y(0) = -0.9$.

$y \longrightarrow 0$ as $t \longrightarrow \infty$

7. $\boxed{16}$ Use the substitution $z = y^{-1}$ to find an explicit general solution to the Bernoulli equation

$$y' + x^{-1}y = xy^2.$$

$$y^{-2}y' + x^{-1}y^{-1} = x$$

Let $z = y^{-1}$ $\quad \dfrac{dz}{dx} = -y^{-2}\dfrac{dy}{dx}$

$$z' - \frac{1}{x}z = -x$$

$$u(x) = e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = \frac{1}{x}$$

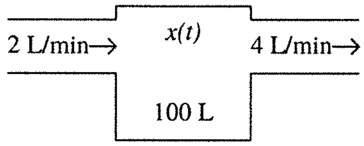
$$\left(\frac{1}{x}z\right)' = -1$$

$$\frac{1}{x}z = C - x$$

$$z = x(C-x)$$

$$y = \frac{1}{x(C-x)}$$

8. $\boxed{16}$ A tank is initially filled with 100 L of a brine solution with a concentration of 0.2 kg/L. A solution containing 0.1 kg of salt per liter is being pumped into the tank at a rate at 2 L/min. The tank is well-stirred (via mathematical magic) and drains at 4 L/min.



The amount of salt $x(t)$ (in kg) in the tank at time t (in min) is modeled by the initial value problem

$$\frac{dx}{dt} = 0.2 - \frac{4x}{100-2t}, \qquad x(0) = 20.$$

Find the amount of salt in the tank for $t \in [0, 50)$, i.e., solve the initial value problem.

$$\frac{dx}{dt} - \left(\frac{-2}{50-t}\right) x = 0.2 \qquad u(t) = e^{\int \frac{2}{50-t}\,dt} = e^{-2 \ln|50-t|} = \frac{1}{(50-t)^2}$$

$$\frac{1}{(50-t)^2} x = \int 0.2\,(50-t)^{-2}\,dt$$

$$= 0.2\,(50-t)^{-1} + C$$

$$x = 0.2\,(50-t) + C\,(50-t)^2$$

$x(0) = 20$

$$20 = 10 + C\,(50)^2 \quad \text{so} \quad C = \frac{10}{50^2}$$

$$x = 0.2\,(50-t) + \frac{10}{50^2}\,(50-t)^2$$

Page	1	2	3	4	5	6	Total
Value	12	12	20	24	16	16	100
Points							