

Math 274 Using Formulas

Sections: 4.1-4.6

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Reduction of Order

If $y_1(t)$ is a solution, not identically zero, to $y'' + p(t)y' + q(t)y = 0$ on I , then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) dt}}{(y_1(t))^2} dt$$

is a second, linearly independent solution.

1. The equation

$$y'' - 2by' + b^2y = 0 \quad (1)$$

has a repeated Characteristic Root of $\lambda = b$ and hence $y_1 = e^{bt}$ is a solution to (1). Use Reduction of Order to show that a second solution to (1) is given by $y_2 = te^{bt}$.

$$y_2 = e^{bt} \int \frac{e^{-\int -2b dt}}{(e^{bt})^2} dt = e^{bt} \int \frac{e^{2bt}}{e^{2bt}} dt = e^{bt} \int dt = te^{bt}$$

2. Use the Wronskian to verify that the solution found via Reduction of Order, y_2 , is linearly independent from y_1 .

$$\begin{aligned} W[y_1, y_2] &= \det \begin{bmatrix} y_1 & y_1 \int \frac{e^{-\int p(t) dt}}{y_1^2} \\ y_1' & y_1' \int \frac{e^{-\int p(t) dt}}{y_1^2} + y_1 \cdot \frac{e^{-\int p(t) dt}}{y_1^2} \end{bmatrix} = y_1 y_1' \int \frac{e^{-\int p(t) dt}}{y_1^2} + e^{-\int p(t) dt} - y_1 y_1' \int \frac{e^{-\int p(t) dt}}{y_1^2} \\ &= e^{-\int p(t) dt} \neq 0 \end{aligned}$$

3. Find a general solution for

$$ty'' - (t+1)y' + y = 0$$

provided that $y_1 = e^t$ is a solution.

std form: $y'' + \left(-1 - \frac{1}{t}\right)y' + \frac{1}{t}y = 0$

$$y_2 = e^t \int \frac{e^{\int (1+\frac{1}{t}) dt}}{e^{2t}} dt = e^t \int \frac{e^t \cdot t}{e^{2t}} dt = e^t \int t e^{-t} dt = e^t \left[-t e^{-t} - e^{-t} \right]$$

$u=t \quad dv=e^{-t} dt$
 $du=dt \quad v=-e^{-t}$

so $y_2 = -t - 1$ (Note any constant multiple is also a solution, so
 $y = C_1 e^t + C_2 (t+1)$ we could choose to use $y_2 = 1+t$)

Variation of Parameters

If y_1 and y_2 are linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then a particular solution to $y'' + p(t)y' + q(t)y = g(t)$ is given by

$$y_p(t) = y_1(t) \int \frac{-g(t)y_2(t)}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)} dt.$$

4. Given that $y = C_1 t^{-2} + C_2$ is a general solution to

$$t^2 y'' + 3ty' = 0,$$

find a particular solution to

$$t^2 y'' + 3ty' = 4t^{-2} \ln t. \quad \leftarrow \quad y'' + \frac{3}{t} y' = 4t^{-4} \ln t$$

$$\begin{aligned} y_1 &= t^{-2} \\ y_2 &= 1 \\ W[y_1, y_2] &= \det \begin{bmatrix} t^{-2} & 1 \\ -2t^{-3} & 0 \end{bmatrix} = 2t^{-3} \\ y_p &= t^{-2} \int \frac{(-4t^{-4} \ln t)}{2t^{-3}} dt + \int \frac{(4t^{-4} \ln t)t^{-2}}{2t^{-3}} dt = 2t^{-2} \int \frac{\ln t}{t} dt + \int 2t^{-3} \ln t dt \\ &\quad u = \ln t \quad dv = 2t^{-3} dt \\ &\quad du = \frac{1}{t} dt \quad v = -t^{-2} \\ &= -2t^{-2} \cdot \frac{1}{2} (\ln t)^2 - t^{-2} \ln t - \frac{t^{-2}}{2} \end{aligned}$$

5. See problem 3, then find a general solution for

$$\begin{aligned} ty'' - (t+1)y' + y &= t^2. \quad \leftarrow \quad y'' + (-\frac{1}{t})y' + \frac{1}{t^2}y = t^2 \\ y_1 &= e^t \\ y_2 &= -t^{-1} \\ W[y_1, y_2] &= \det \begin{bmatrix} e^t & -t^{-1} \\ e^t & -1 \end{bmatrix} = -e^t + te^t + e^t = te^t \end{aligned}$$

$$\begin{aligned} y_p &= e^t \int \frac{-t(-t-1)}{te^t} dt + (-t-1) \int \frac{te^t}{te^t} dt = e^t \int (t+1)e^t dt - (t+1)t \\ &\quad u = t+1 \quad dv = e^t dt \\ &\quad du = dt \quad v = e^t \\ &= e^t \left((t+1)(-e^{-t}) - e^{-t} \right) - t^2 - t \end{aligned}$$

$$= -(t+1) - 1 - t^2 - t = -t^2 - 2t - 2$$

$$y = C_1 e^t + C_2 (-t-1) - t^2 - 2t - 2 \quad \text{or} \quad y = C_1 e^t + C_2 (t+1) - t^2$$