

1. Use the Real Method of Undetermined Coefficients to find a general solution for each of the following.

(a) $y'' - y' - 2y = 4e^{3t} - 4t$ $(\lambda - 2)(\lambda + 1) = 0 \quad \lambda_1 = 2, \lambda_2 = -1$

$$\begin{aligned} y_p &= Ae^{3t} + Bt + C & y_p'' - y_p' - 2y_p &= 9Ae^{3t} - 3Ae^{3t} - B - 2Ae^{3t} - 2Bt - 2C \\ y_p' &= 3Ae^{3t} + B & &= 4Ae^{3t} - 2Bt - B - 2C \\ y_p'' &= 9Ae^{3t} & &= 4e^{3t} - 4t \quad \text{so } A = 1, B = 2, C = -1 \end{aligned}$$

$$y = C_1 e^{-t} + C_2 e^{3t} + e^{3t} + 2t - 1$$

(b) $y'' - y' - 2y = 20 \cos 2t$ $20 \cos 2t = -4A \cos 2t - 4B \sin 2t$

$$\begin{aligned} y_p &= A \cos 2t + B \sin 2t & + 2A \sin 2t - 2B \cos 2t \\ y_p' &= -2A \sin 2t + 2B \cos 2t & - 2A \cos 2t - 2B \sin 2t \\ y_p'' &= -4A \cos 2t - 4B \sin 2t & = \cos 2t [-6A - 2B] + \sin 2t [2A - 6B] \end{aligned}$$

$$\begin{array}{rcl} \cos 2t: & -6A - 2B = 20 & -3A - B = 10 \\ \sin 2t: & 2A - 6B = 0 & \hline 3A - 9B = 0 \\ & & -10B = 10 \end{array}$$

$$y = C_1 e^{2t} + C_2 e^{-t} - 3 \cos 2t - \sin 2t \quad B = -1, A = -3$$

(c) $y'' - y' - 2y = 18e^{2t}$

$$y_p = Ate^{2t} \quad 18e^{2t} = [4A + 4At]e^{2t} + [-A - 2At]e^{2t} + [-2At]e^{2t}$$

$$y_p' = (A + 2At)e^{2t} \quad = 3Ae^{2t} \quad \text{so } A = 6$$

$$y_p'' = (2A + 4At)e^{2t}$$

$$y = C_1 e^{2t} + C_2 e^{-t} + 6te^{2t}$$

2. [4] Use the Real Method of Undetermined Coefficients to find the solution to the initial value problem

$$y'' + 4y = \sin 2t, \quad y(0) = 2, y'(0) = 0. \quad \lambda = \pm 2i$$

$$y_p = A + \cos 2t + Bt \sin 2t$$

$$y_p' = (A + 2Bt) \cos 2t + (B - 2At) \sin 2t$$

$$y_p'' = (4B - 4At) \cos 2t + (-4A - 4Bt) \sin 2t$$

$$\begin{aligned} y_p'' + 4y_p &= [4B - 4At] \cos 2t + [-4A - 4Bt] \sin 2t + 4A + \cos 2t + 4Bt \sin 2t \\ &= 4B \cos 2t - 4A \sin 2t \\ &= \sin 2t \quad \Rightarrow \quad B = 0, \quad A = -\frac{1}{4} \end{aligned}$$

$$y_p = -\frac{1}{4}t \cos 2t$$

$$y = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{4}t \cos 2t$$

$$y(0) = 2 \Rightarrow C_1 = 2$$

$$y' = -4 \sin 2t + 2C_2 \cos 2t - \frac{1}{4} \cos 2t + \frac{1}{2}t \sin 2t$$

$$y'(0) = 2C_2 - \frac{1}{4} = 0 \Rightarrow C_2 = \frac{1}{8}$$

$$y = 2 \cos 2t + \frac{1}{8} \sin 2t - \frac{t}{4} \cos 2t$$

3. Use the Complex Method of Undetermined Coefficients to find a general solution for each of the following.

(a) [3] $y'' - y' - 2y = 20 \sin 2t = \text{Im} (20 e^{2ti})$

$$z_p = Ae^{2ti}$$

$$z_p' = 2iAe^{2ti}$$

$$z_p'' = -4Ae^{2ti}$$

$$z_p = (-3+i)e^{2ti}$$

$$y_p = \text{Im}(z_p)$$

$$= -3 \sin 2t + \cos 2t$$

$$\begin{aligned} z_p'' - z_p' - 2z_p &= (-4A - 2iA - 2A)e^{2ti} \\ &= 20e^{2ti} \end{aligned}$$

$$\text{so } A = \frac{20}{-6-2i} = \frac{10}{-3-i} \cdot \frac{(-3+i)}{(-3+i)} = \frac{10(-3+i)}{10} = -3+i$$

$$y = C_1 e^{-t} + C_2 e^{2t} + \cos 2t - 3 \sin 2t$$

(b) [4] $y'' + 4y = \cos 2t = \text{Re}(e^{2it})$

$$z_p = Ate^{2it}$$

$$z_p' = (A + 2iAt)e^{2it}$$

$$z_p'' = (4Ai - 4At)e^{2it}$$

$$z_p'' + 4z_p = (4Ai - 4At)e^{2it} + 4At e^{2it}$$

$$= 4Aie^{2it}$$

$$y_p = \text{Re}\left(\frac{-i}{4} te^{2it}\right)$$

$$= e^{2it}$$

$$= \frac{t}{4} \sin 2t$$

$$\text{so } A = \frac{1}{4i} = \frac{-i}{4}$$

$$y = C_1 \cos 2t + C_2 \sin 2t + \frac{t}{4} \sin 2t$$