Math 274 Exam 3 **Section:** ______ **Name:** ____________________

7 Nov 2018 Point values in $\boxed{\text{boxes}}$.

1. $\boxed{4}$ Compute the matrix product.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $\boxed{4}$ For a piecewise continuous function $f(t)$ of exponential order defined on $t \in [0, \infty)$, state the definition of the Laplace transform of $f(t)$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

3. $\boxed{8}$ Find the inverse Laplace transform of

$$F(s) = \frac{2se^{-s}}{s^2 + 6s + 25}.$$

$$= e^{-s} \left[\frac{2(s+3) - 6}{(s+3)^2 + 4^2} \right]$$

$$f(t) = H(t-1) \left[2e^{-3(t-1)} \cos(4(t-1)) - \frac{3}{2} e^{-3(t-1)} \sin(4(t-1)) \right]$$

4. $\boxed{10}$ Find the Laplace transform of the following.

$$g(t) = \begin{cases} t\sin 2t, & t < \pi \\ 0, & \pi < t \end{cases}$$

$g(t) = t \sin 2t - H(t-\pi)(t \sin 2t)$

$G(s) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] - e^{-\pi s}\mathcal{L}\{(t+\pi)\sin(2t+2\pi)\}$

$= \frac{4s}{(s^2+4)^2} - e^{-\pi s}\left[\frac{4s}{(s^2+4)^2} + \frac{2\pi}{(s^2+4)^2}\right]$

5. $\boxed{10}$ Applying the Laplace transform to the initial value problem

$$y'' + y' - 2y = -6e^t, \qquad y(0) = 4, y'(0) = -7$$

gives the following

$$Y(s) = \frac{4s^2 - 7s - 3}{(s-1)^2(s+2)}. \tag{1}$$

Determine $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, the solution to the given initial value problem.

$\frac{4s^2-7s-3}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$4s^2-7s-3 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

$y(t) = e^t - 2te^t + 3e^{-2t}$

$s=1$: $-6 = 3B$ so $B=-2$

$s=-2$: $27 = 9C$ so $C=3$

s^2 : $4 = A+C$ so $A=1$

6. $\boxed{16}$ Solve the initial value problem.

$$y'' + 4y = f(t), \qquad y(0) = 1, y'(0) = 0$$

where

$$f(t) = \begin{cases} 8e^{2t}, & t < 1 \\ 0, & 1 < t \end{cases} = 8e^{2t} - H(t-1)\, 8e^{2t}$$

You will find it useful to know that

$$\frac{8}{(s-2)(s^2+4)} = \frac{1}{s-2} - \frac{s+2}{s^2+4}.$$

$$s^2 Y - s + 4Y = \frac{8}{s-2} - e^{-s} \mathcal{L}\{8e^{2t+2}\}$$

$$= \frac{8}{s-2} - e^2 \cdot e^{-s} \left[\frac{8}{s-2}\right]$$

$$Y = \frac{8}{(s-2)(s^2+4)} - e^2 \cdot e^{-s} \left[\frac{8}{(s-2)(s^2+4)}\right] + \frac{s}{s^2+4}$$

$$= \left[\frac{1}{s-2} - \cancel{\frac{s}{s^2+4}} - \frac{2}{s^2+4}\right] - e^2 \cdot e^{-s} \left[\frac{1}{s-2} - \frac{s}{s^2+4} - \frac{2}{s^2+4}\right] + \cancel{\frac{s}{s^2+4}}$$

$$y = e^{2t} - \sin 2t - e^2 H(t-1) \left[e^{2(t-1)} - \cos(2t-2) - \sin(2t-2)\right]$$

7. $\boxed{4}$ The problem on this page and the next both involve the same rational expression. Perform the partial fraction decomposition on the following.

$$\frac{1}{s(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \qquad \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$1 = A(s+1) + Bs$$

$s = 0 \Rightarrow A = 1$

$s = -1 \Rightarrow B = -1$

8. Assume $g(t)$ is piecewise continuous and of exponential order and consider the initial value problem

$$y'' + y' = g(t), \qquad y(0) = 1, y'(0) = 0.$$

(a) $\boxed{8}$ Find the solution. Express your solution in terms of a convolution.

$$s^2 Y - s + sY - 1 = G(s)$$

$$Y(s(s+1)) = G(s) + s + 1$$

$$Y = G(s)\left[\frac{1}{s} - \frac{1}{s+1}\right] + \frac{1}{s}$$

$$y = g(t) * (1 - e^{-t}) + 1$$

(b) $\boxed{4}$ Express the convolution in (a) in terms of the appropriate integral. **Do not evaluate the integral.**

$$y = 1 + \int_0^t g(u)\left(1 - e^{-(t-u)}\right) du$$

OR

$$1 + \int_0^t g(t-u)(1 - e^{-u})\, du$$

9. $\boxed{16}$ Let $f(t)$ be the 2-periodic function with one period given by $f(t) = \begin{cases} 1, & t < 1 \\ 0, & 1 \le t < 2 \end{cases}$.
Applying the Laplace transform to the initial value problem

$$y' + y = f(t), \qquad y(0) = 0$$

gives the following

$$Y(s) = \frac{1}{s(s+1)(1+e^{-s})}.$$

Determine $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, the solution to the given initial value problem. Express your solution as a piecewise defined function for $t \in [0,4)$.

$$Y(s) = \frac{1}{s(s+1)} \cdot \frac{1}{1+e^{-s}}$$

$$= \left[\frac{1}{s} - \frac{1}{s+1}\right] \sum_{n=0}^{\infty} (-1)^n e^{-sn}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{s} - \frac{1}{s+1}\right] (-1)^n e^{-sn}$$

$$y(t) = \sum_{n=0}^{\infty} \left(1 - e^{-(t-n)}\right)(-1)^n H(t-n)$$

$$= 1 - e^{-t} - H(t-1)\left[1 - e^{-t+1}\right] + H(t-2)\left[1 - e^{-t+2}\right]$$

$$- H(t-3)\left[1 - e^{-t+3}\right] + \cdots$$

$$y(t) = \begin{cases} 1 - e^{-t}, & \text{for } 0 \le t < 1 \\ e^{1-t} - e^{-t}, & \text{for } 1 \le t < 2 \\ 1 - e^{2-t} + e^{1-t} - e^{-t}, & \text{for } 2 \le t < 3 \\ e^{3-t} - e^{2-t} + e^{1-t} - e^{-t}, & \text{for } 3 \le t < 4 \\ \text{etc.}, & \text{etc.} \end{cases}$$

10. $\boxed{6}$ Solve the initial value problem

$$y'' + 4y = 0, \qquad y(0) = 0, y'(0) = 2.$$

$s^2 Y - 2 + 4Y = 0$

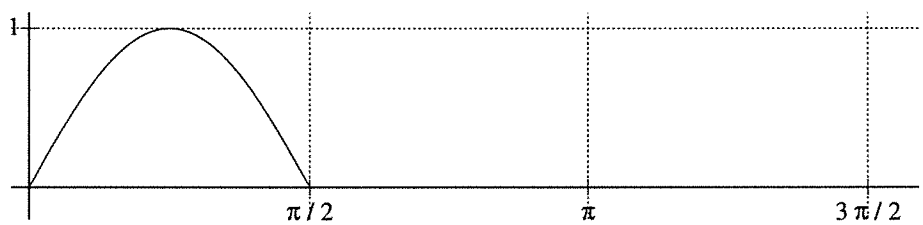
$y(t) = \sin 2t$

$Y = \frac{2}{s^2+4}$

11. $\boxed{4}$ Find a function $g(t)$ so that the solution to the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = 0, y'(0) = 2.$$

has the following graph.

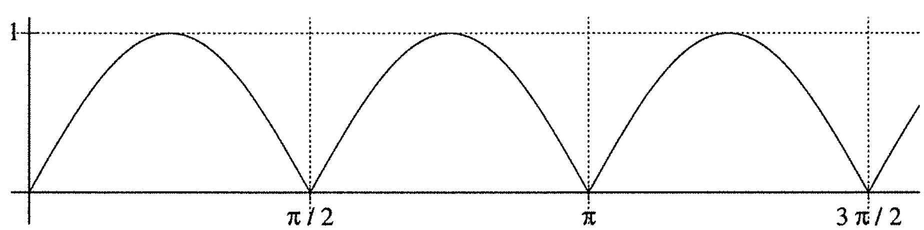


$g(t) = 2\delta\left(t - \frac{\pi}{2}\right)$

12. $\boxed{6}$ Find a function $f(t)$ so that the solution to the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, y'(0) = 2.$$

is periodic with the following graph.



$f(t) = \sum_{n=1}^{\infty} 4\delta\left(t - \frac{n\pi}{2}\right)$

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