

1. Find the form of a particular solution suggested by the Method of Undetermined Coefficients.

(a) $y'' + 2y' + y = 3te^{2t} + 3t^2$
 $y_1 = e^{-t}$
 $y_2 = te^{-t}$
 solve
 $y'' + 2y' + y = 0$
 $r^2 + 2r + 1 = 0$
 $(r+1)^2 = 0$

$$y_p = (At+B)e^{2t} + (Ct^2+Dt+E)$$

(b) $y'' + 2y' + y = 4t \sin t$
 $y_1 = e^{-t} \cos 3t$
 $y_2 = e^{-t} \sin 3t$
 solve
 $y'' + 2y' + 10y = 0$

$$y_p = (At+B) \cos t + (Ct+D) \sin t$$

(c) $y'' + 2y' + y = 4te^{-t}$
 $y_1 = e^{-t} \cos 3t$
 $y_2 = e^{-t} \sin 3t$
 solve
 $y'' + 2y' + 10y = 0$

$$y_p = t^2 (At+B)e^{-t}$$

(d) $y'' + 2y' + 10y = 6e^{-t} \cos 3t$
 $y_1 = e^{-t} \cos 3t$
 $y_2 = e^{-t} \sin 3t$
 solve
 $y'' + 2y' + 10y = 0$

$$(y+1)^2 = -9$$

$$y_p = Ate^{-t} \cos 3t + Bte^{-t} \sin 3t$$

$$y = -1 \pm 3i$$

2. Find the form of a particular solution suggested by the Method of Undetermined Coefficients in the complex plane.

(a) $y'' + 2y' + y = 4te^{it}$ $y_p = (At+B)e^{it}$

(b) $y'' + 2y' + 10y = 6e^{(-1+3i)t}$
 $y_p = A + e^{(-1+3i)t}$

On separate sheets of paper stapled to this page do the following.

3. Consider the equation

$$y'' + 2y' + y = 4t \sin t. \quad (1)$$

- (a) 1 Find the general solution to $y'' + 2y' + y = 0$. $y = C_1 e^{-t} + C_2 te^{-t}$
 (b) 4 Use the Real Method of Undetermined Coefficients to find a particular solution to (1).
 (c) 4 Use the Complex Method of Undetermined Coefficients to find a particular solution to (1).
 (d) 1 Find the general solution to (1). $y = C_1 e^{-t} + C_2 te^{-t} + (-2t+2) \cos t + 2 \sin t$

4. Consider the equation

$$y'' + 2y' + 10y = 6e^{-t} \cos 3t. \quad (2)$$

- (a) 1 Find the general solution to $y'' + 2y' + 10y = 0$. $y = C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t$
 (b) 4 Use the Real Method of Undetermined Coefficients to find a particular solution to (2).
 (c) 4 Use the Complex Method of Undetermined Coefficients to find a particular solution to (2).
 (d) 1 Find the general solution to (2). $y = C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t + e^{-t} t \sin 3t$

$$3. \quad y'' + 2y' + y = 4t \sin t \quad (1)$$

$$a) \quad y = C_1 e^{-t} + C_2 t e^{-t}$$

$$b) \quad y_p = (At+B) \cos t + (Ct+D) \sin t$$

$$y_p' = (A+Ct+D) \cos t + (C-At-B) \sin t$$

$$y_p'' = (2C-At-B) \cos t + (-2A-Ct-D) \sin t$$

Substituting gives

$$4t \sin t = [2Ct + 2A + 2C + 2D] \cos t + [-2At - 2A - 2B - 2C] \sin t.$$

Equating coefficients gives

$$\underline{t \cos t}: \quad 0 = 2C \quad \text{so} \quad C = 0$$

$$\underline{\cos t}: \quad 0 = 2A + 2C + 2D \quad \text{so} \quad D = 2$$

$$\underline{t \sin t}: \quad 4 = -2A \quad \text{so} \quad A = -2$$

$$\underline{\sin t}: \quad 0 = -2A - 2B + 2C \quad \text{so} \quad B = 2$$

so $y_p = (-2t + 2) \cos t + 2 \sin t$ is a particular solution.

c) Since $4t \sin t = \operatorname{Im}(4te^{it})$ we look for a solution to $z'' + 2z' + z = 4te^{it}$.

$$z_p = (At+B)e^{it}$$

$$z_p' = (A+i(At+B))e^{it}$$

$$z_p'' = (2Ai - (At+B))e^{it}$$

Substituting gives

$$4te^{it} = (2Ai + 2A + 2Ait + 2iB)e^{it}.$$

Equating coefficients gives

$$\underline{te^{it}}: \quad 4 = 2Ai \quad \text{so} \quad A = -2i$$

$$\underline{e^{it}}: \quad 0 = 2A(1+i) + 2iB$$

$$\text{so} \quad 4i(1+i) = 2iB \Rightarrow B = 2 + 2i$$

$$y_p = \operatorname{Im}(z_p) = \operatorname{Im} \left[(-2it + 2 + 2i)(\cos t + i \sin t) \right] = -2t \cos t + 2 \sin t + 2 \cos t$$

D) $y = C_1 e^{-t} + C_2 t e^{-t} + (-2t + 2) \cos t + 2 \sin t$ solves (1).

$$4. \quad y'' + 2y' + 10y = 6e^{-t} \cos 3t \quad (2)$$

A) $y_p = C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t$

B) $y_p = Ate^{-t} \cos 3t + Bte^{-t} \sin 3t$

$$y_p' = [-At + 3Bt + A] e^{-t} \cos 3t + [-3At - Bt + B] e^{-t} \sin 3t$$

$$y_p'' = [-8At - 6Bt - 2A + 6B] e^{-t} \cos 3t + [6At - 8Bt - 6A - 2B] e^{-t} \sin 3t$$

Substituting & simplifying gives

$$y_p'' + 2y_p' + 10y_p = 6Be^{-t} \cos 3t - 6Ae^{-t} \sin 3t = 6e^{-t} \cos 3t$$

so $B=1, A=0$

$$y_p = te^{-t} \sin 3t \text{ solve (2)}$$

Note: There is a lot of algebraic simplification not shown.

C) Since $6e^{-t} \cos 3t = \operatorname{Re}(6e^{(-1+3i)t})$ we look for a solution to $z'' + 2z' + 10z = 6e^{(-1+3i)t}$

Let $r = -1 + 3i$.

$$z_p = Ate^{rt}$$

$$z_p' = A(1+tr)e^{rt}$$

$$z_p'' = (2Ar + Atr^2)e^{rt}$$

Substituting gives

$$z_p'' + 2z_p' + 10z_p = [2Ar + Atr^2 + 2A + 2Atr + 10At] e^{rt}$$

$$= [At(\underbrace{r^2 + 2r + 10}_{=0}) + 2A(r+1)] e^{rt} \quad \text{Note: } r \text{ solves } r^2 + 2r + 10 = 0$$

$$= 6e^{rt} \quad \text{so} \quad A = \frac{6}{2(r+1)} = \frac{3}{3i} = -i$$

$$y_p = \operatorname{Re}(z_p) = \operatorname{Re}(-it e^{-t} (\cos 3t + i \sin 3t)) = te^{-t} \sin 3t$$

D) $y = C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t + te^{-t} \sin 3t$