Exam 1 - Thursday, 1 February 2018

Show all appropriate work to receive full credit. Point values are in boxes.

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at Montana State University, and that I have neither given nor received any unauthorized assistance (calculator, cell phone, notes, other people’s exams, etc.) on this exam. Furthermore, I have and will not discuss the content of this exam, or provide any written or verbal aid to others. I understand that if I violate this statement, I will receive an exam score of 0."

SIGNATURE: ________________________________

1. [4] Evaluate

\[ \text{Re}((2 - i)(3 - i)). \]

\[ \text{Re} (6 - 5i + i^2) = 5 \]

2. The direction field for \( \frac{dy}{dx} = f(x, y) \) is shown below. Use it to answer the following.

(a) [2] Classify the differential equation as \textbf{Autonomous} or \textbf{Not Autonomous}. Circle your choice.

(b) [6] Sketch the solution curves, i.e. the trajectories, satisfying the following initial data.

i. \( y(0) = 1 \)

ii. \( y(2) = 0 \)

iii. \( y(-1) = 3 \)
3. Consider the autonomous differential equation

\[ \frac{dy}{dt} = f(y). \]

The only thing that is known about \( f(y) \) is that the graph of \( f(y) \) is given below.

(a) Sketch the phase line for the differential equation and classify each equilibrium as a sink, source, or node.

(b) Use the phase line to predict the asymptotic behavior as \( t \to \infty \) of the solution satisfying \( y(0) = 0.9 \).

\[ y \to -2 \]

(c) Use the phase line to predict the asymptotic behavior as \( t \to \infty \) of the solution satisfying \( y(0) = 1 \).

\[ y \to 1 \]
4. Consider the equation

\[(y - xy^2)dx + dy = 0.\]  

(1)

(a) Show equation (1) is not exact.

\[
\frac{\partial}{\partial y} (y - xy^2) = 1 - 2xy \neq \frac{\partial}{\partial x} \quad (1)
\]

(b) Equation (1) can be rewritten in the form

\[\frac{dy}{dx} + y = xy^2.\]  

(2)

Use an appropriate substitution to find an explicit general solution to (2).

\[y^{-2} \frac{dy}{dx} + y^{-1} = x\]

\[\mu(x) = e^{-x}\]

\[\int e^{-x} \, (-x) \, dx = e^{x} \left[ x e^{-x} + e^{-x} + C \right] = x + C e^x\]

\[u = x \quad dv = -e^{-x} \, dx \quad \frac{dv}{dx} = e^{-x}\]

\[y = \frac{1}{x + 1 + Ce^x} \quad \text{or} \quad y = 0\]
5. The following differential equation is exact

\[(\cos \theta) \, dr - (r \sin \theta - e^\theta) \, d\theta = 0.\]

Find an explicit general solution of the form \(r = f(\theta)\).

\[r \cos \theta + e^\theta = C \]

so

\[r = (C - e^\theta) \sec \theta.\]

Sec 9.2.4

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6. For \(x, t > 0\), use an appropriate substitution to find an implicit general solution to the homogeneous equation

\[\frac{dx}{dt} = \frac{x^2 + t\sqrt{t^2 + x^2}}{tx}.\]

\[\frac{dx}{dt} = \frac{\sqrt{x^2}}{t} + \sqrt{\left(\frac{t}{x}\right)^2 + 1}\]

\[v + \frac{1}{t} \frac{dv}{dt} = v + \sqrt{\frac{1}{v^2} + 1}\]

\[t \frac{dv}{dt} = \sqrt{\frac{1 + v^2}{v^2}}\]

\[
\int \frac{v}{\sqrt{1 + v^2}} \, dv = \int \frac{d\frac{1}{t}}{t}
\]

\[\sqrt{1 + \left(\frac{y}{x}\right)^2} = \ln \left| \frac{1}{t} \right| + C\]
7. A balloon is initially filled with 1 L of helium. A tank with a mix of 50% helium and 50% oxygen is being used to fill the balloon at a rate of 0.2 L/min. The seal between the tank and the balloon is not airtight and the resulting mixture inside the balloon (assume the gas inside the balloon is well mixed) is escaping at a rate of 0.1 L/min. The balloon pops when it reaches a volume of 2 L. For \( t \in [0, 10] \) the amount of helium, in L, in the the balloon is modeled by the initial value problem

\[
\frac{dh}{dt} = \frac{1}{10} - \frac{h}{1 + t/10} \cdot \frac{1}{10}, \quad h(0) = 1.
\]

(a) \( \boxed{15} \) Find the amount of helium in the balloon, in L, as a function of time for \( t \in [0, 10] \), i.e. solve the initial value problem.

\[
\frac{dh}{dt} + \frac{h}{10 + t} = \frac{1}{10}, \quad m(t) = 10 + t
\]

\[
h = \frac{1}{10 + t} \int \left(1 + \frac{t}{10}\right) \, dt
\]

\[
= \frac{1}{10 + t} \left( t + \frac{t^2}{20} + C \right)
\]

\( h(0) = 1 \quad \Rightarrow \quad C = 10 \)

\[
h(t) = \frac{t + \frac{t^2}{20} + 10}{t + 10}
\]

(b) \( \boxed{3} \) Find the amount of helium in the balloon, in L, immediately prior to it popping, i.e. evaluate the following limit.

\[
\lim_{t \to 10} h(t) = \frac{25}{20} = 1.25
\]
8. Consider the initial value problem

\[ y' = -\sqrt{y}, \quad y(2) = 1. \]

(a) \[4\] Does the Existence/Uniqueness Theorem apply to this initial value problem? Justify your response.

Since \[ -\frac{1}{2\sqrt{y}} \neq \frac{3}{3y} \left( -\sqrt{y} \right) \] are continuous near \( y = 1 \),

the theorem applies.

Wolfram Alpha says the solutions to this initial value problem are

\[ y_1(x) = \frac{x^2}{4}, \quad \text{and} \quad y_2(x) = \frac{(x-4)^2}{4}. \]

(b) \[1\] In light of your response to (a), what can you conclude about Wolfram Alpha’s solutions?

At least one of them is wrong since a unique solution exists.

(c) \[3\] Verify \( y_1(x) \) and \( y_2(x) \) both satisfy the initial data, i.e. \( y_1(2) = 1 \) and \( y_2(2) = 1 \).

\[ y_1(2) = \frac{4}{4} = 1 \quad \checkmark \quad y_2(2) = \frac{(2-4)^2}{4} = 1 \quad \checkmark \]

(d) \[3\] Verify \( y_1 \) does not satisfy the differential equation near the initial data, i.e. \( y_1' \neq -\sqrt{y_1} \) near \( x = 2 \).

[HINT: \( \sqrt{x^2} = |x| \). Choose the sign appropriately.]

\[ y_1' = \frac{x}{2} \quad -\sqrt{y_1} = -\sqrt{\frac{x^2}{4}} = -\left| \frac{x}{2} \right| = -\frac{x}{2} \quad \neq \quad \frac{x}{2} \]

(e) \[3\] Verify \( y_2 \) satisfies the differential equation near the initial data, i.e. \( y_2' = -\sqrt{y_2} \) near \( x = 2 \).

[HINT: \( \sqrt{x^2} = |x| \). Choose the sign appropriately.]

\[ y_2' = \frac{x-4}{2} \]

\[ -\sqrt{y_2} = -\sqrt{\frac{(x-4)^2}{4}} = -\left| \frac{x-4}{2} \right| = -\frac{x-4}{2} \quad \neq \quad \frac{x-4}{2} \quad \checkmark \]

So \( y_2' = -\sqrt{y_2} \).