Math 274 Section:_____ Name:_________________

Exam 1 - Thursday, 1 February 2018
Show all appropriate work to receive full credit. Point values are in $\boxed{\text{boxes}}$
HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at Montana State University, and that I have neither given nor received any unauthorized assistance (calculator, cell phone, notes, other people's exams, etc.) on this exam. Furthermore, I have and will not discuss the content of this exam, or provide any written or verbal aid to others. I understand that if I violate this statement, I will receive an exam score of 0."

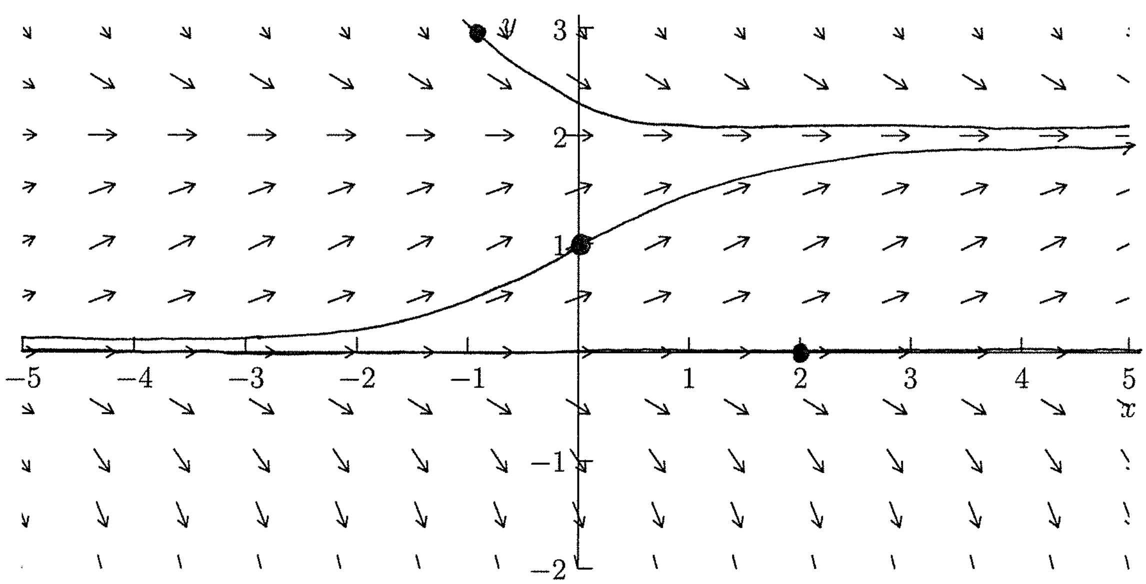
SIGNATURE:______________________________

1. $\boxed{4}$ Evaluate

$$\mathrm{Re}\left((2-i)(3-i)\right).$$

$$Re\left(6-5i+i^2\right)=5$$

2. The direction field for $\frac{dy}{dx}=f(x,y)$ is shown below. Use it to answer the following.



(a) $\boxed{2}$ Classify the differential equation as **Autonomous** or **Not Autonomous**. Circle your choice.

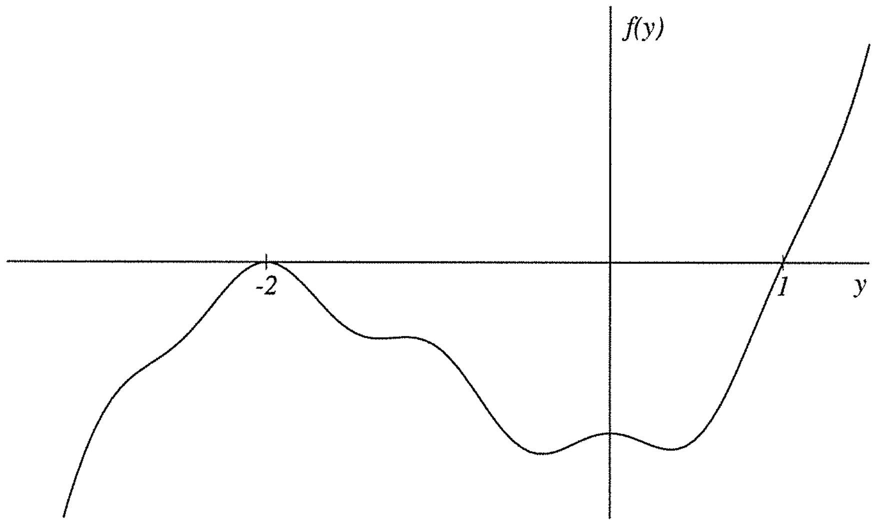
(b) $\boxed{6}$ Sketch the solution curves, i.e. the trajectories, satisfying the following initial data.

i. $y(0)=1$ ii. $y(2)=0$ iii. $y(-1)=3$

3. Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y).$$

The only thing that is known about $f(y)$ is that the graph of $f(y)$ is given below.



(a) $\boxed{7}$ Sketch the phase line for the differential equation and classify each equilibria as a sink, source, or node.

$y = 1$, source

$y = -2$, node

(b) $\boxed{2}$ Use the phase line to predict the asymptotic behavior as $t \to \infty$ of the solution satisfying $y(0) = 0.9$.

$y \longrightarrow -2$

(c) $\boxed{2}$ Use the phase line to predict the asymptotic behavior as $t \to \infty$ of the solution satisfying $y(0) = 1$.

$y \equiv 1$

4. Consider the equation

$$(y - xy^2)dx + dy = 0. \qquad (1)$$

(a) $\boxed{5}$ Show equation (1) is not exact.

$$\frac{\partial}{\partial y}(y - xy^2) = 1 - 2xy \neq 0 = \frac{\partial}{\partial x}(1)$$

(b) $\boxed{15}$ Equation (1) can be rewritten in the form

$$\frac{dy}{dx} + y = xy^2. \qquad (2)$$

Use an appropriate substitution to find an explicit general solution to (2).

$$y^{-2}\frac{dy}{dx} + y^{-1} = x \qquad \text{Let } v = y^{-1}$$

$$-\frac{dv}{dx} + v = x \qquad \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\frac{dv}{dx} - v = -x \qquad \mu(x) = e^{-x}$$

$$v = e^x \int e^{-x}(-x)\,dx = e^x\left[xe^{-x} + e^{-x} + C\right] = x + 1 + Ce^x$$

$$u = x \qquad dv = -e^{-x}\,dx$$

$$du = dx \qquad v = e^{-x}$$

$$y = \frac{1}{x + 1 + Ce^x} \quad \text{OR} \quad y \equiv 0$$

5. $\boxed{10}$ The following differential equation is exact

$$(\cos\theta)\,dr - (r\sin\theta - e^{\theta})d\theta = 0.$$

Find an explicit general solution of the form $r = f(\theta)$.

$$r\cos\theta + e^{\theta} = C$$

See §2.4

15

so

$$r = (C - e^{\theta})\sec\theta$$

6. $\boxed{15}$ For $x, t > 0$, use an appropriate substitution to find an implicit general solution to the homogeneous equation

$$\frac{dx}{dt} = \frac{x^2 + t\sqrt{t^2+x^2}}{tx}.$$

$$\frac{dx}{dt} = \frac{x}{t} + \sqrt{\left(\frac{t}{x}\right)^2 + 1}$$

Let $v = \frac{x}{t}$ so $tv = x$

$$v + t\frac{dv}{dt} = \frac{dx}{dt}$$

$$v + t\frac{dv}{dt} = v + \sqrt{\frac{1}{v^2} + 1}$$

$$t\frac{dv}{dt} = \sqrt{\frac{1+v^2}{v^2}}$$

See §2.6

13

$$\int \frac{v}{\sqrt{1+v^2}}\,dv = \int \frac{dt}{t}$$

$$\sqrt{1 + \left(\frac{y}{x}\right)^2} = \ln|t| + C$$

7. A balloon is initially filled with 1 L of helium. A tank with a mix of 50% helium and 50% oxygen is being used to fill the balloon at a rate of 0.2 L/min. The seal between the tank and the balloon is not airtight and the resulting mixture inside the balloon (assume the gas inside the balloon is well mixed) is escaping at a rate of 0.1 L/min. The balloon pops when it reaches a volume of 2 L. For $t \in [0, 10)$ the amount of helium, in L, in the the balloon is modeled by the initial value problem

$$\frac{dh}{dt} = \frac{1}{10} - \frac{h}{1 + t/10} \cdot \frac{1}{10}, \qquad h(0) = 1.$$

(a) $\boxed{15}$ Find the amount of helium in the balloon, in L, as a function of time for $t \in [0, 10)$, i.e. solve the initial value problem.

$$\frac{dh}{dt} + \frac{h}{10 + t} = \frac{1}{10} \qquad \mu(t) = 10 + t$$

$$h = \frac{1}{10 + t} \int \left(1 + \frac{t}{10}\right) dt$$

$$= \frac{1}{10 + t} \left(t + \frac{t^2}{20} + C\right)$$

$$h(0) = 1 \quad \Rightarrow \quad C = 10$$

$$h(t) = \frac{t + t^2/20 + 10}{t + 10}$$

(b) $\boxed{3}$ Find the amount of helium in the balloon, in L, immediately prior to it popping, i.e. evaluate the following limit.

$$\lim_{t \to 10} h(t) = \frac{25}{20} = 1.25$$

8. Consider the initial value problem

$$y' = -\sqrt{y}, \qquad y(2) = 1.$$

(a) $\boxed{4}$ Does the Existence/Uniqueness Theorem apply to this initial value problem? Justify your response.

Since $-\sqrt{y}$ & $\frac{\partial}{\partial y}(-\sqrt{y}) = -\frac{1}{2\sqrt{y}}$ are continuous near $y=1$,

the theorem applies.

Wolfram Alpha says the **solutions** to this initial value problem are

$$y_1(x) = \frac{x^2}{4}, \quad \text{and} \quad y_2(x) = \frac{(x-4)^2}{4}.$$

(b) $\boxed{1}$ In light of your response to (a), what can you conclude about Wolfram Alpha's solutions?

At least one of them is wrong since a unique solution exists.

(c) $\boxed{3}$ Verify $y_1(x)$ and $y_2(x)$ both satisfy the initial data, i.e. $y_1(2) = 1$ and $y_2(2) = 1$.

$y_1(2) = \frac{4}{4} = 1$ ✓ $\qquad y_2(2) = \frac{(-2)^2}{4} = 1$ ✓

(d) $\boxed{3}$ Verify y_1 does not satisfy the differential equation near the initial data, i.e $y_1' \neq -\sqrt{y_1}$ near $x = 2$. [HINT: $\sqrt{\square^2} = |\square|$. Choose the sign appropriately.]

$y_1' = \frac{x}{2}$ $\qquad -\sqrt{y_1} = -\sqrt{\frac{x^2}{4}} = -\left|\frac{x}{2}\right| = -\frac{x}{2}$ for x near 2.

$$\frac{x}{2} \neq -\frac{x}{2}$$

(e) $\boxed{3}$ Verify y_2 satisfies the differential equation near the initial data, i.e $y_2' = -\sqrt{y_2}$ near $x = 2$. [HINT: $\sqrt{\square^2} = |\square|$. Choose the sign appropriately.]

$y_2' = \frac{x-4}{2}$

$-\sqrt{y_2} = -\sqrt{\frac{(x-4)^2}{4}} = -\left|\frac{x-4}{2}\right| = \frac{x-4}{2}$ for x near 2,

so $y_2' = -\sqrt{y_2}$. ✓

Problem	1	2	3	4	5	6	7	8	Total
Value	4	8	11	20	10	15	18	14	100
Points									