

Exam 2 - Thursday, 1 March 2018

Show all appropriate work to receive full credit. Point values are in boxes

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at Montana State University, and that I have neither given nor received any unauthorized assistance (calculator, cell phone, notes, other people's exams, etc.) on this exam. Furthermore, I have and will not discuss the content of this exam, or provide any written or verbal aid to others. I understand that if I violate this statement, I will receive an exam score of 0."

SIGNATURE: _____

1. [12] Find a general solution to the following.

(a) $y'' + 2y' - 3y = 0$

$$(r+3)(r-1) = 0 \quad y = C_1 e^{-3t} + C_2 e^t$$

(b) $4y'' + 4y' + y = 0$

$$y = C_1 e^{-t/2} + C_2 t e^{-t/2}$$

$$(2r+1)^2 = 0$$

$$r = -\frac{1}{2}$$

(c) $x'' + 6x' + 13x = 0$

$$y = C_1 e^{-3t} \cos 2t + C_2 e^{-3t} \sin 2t$$

$$r^2 + 6r + 9 = -4$$

$$(r+3)^2 = -4$$

$$r = -3 \pm 2i$$

2. [8] Find the form of a particular solution suggested by the Method of Undetermined Coefficients for the following. **Do not** solve for the coefficients.

(a) $y'' + 2y' - 3y = te^{-3t} \cos 2t$

$$y_p = (At+B)e^{-3t} \cos 2t + (Ct+D)e^{-3t} \sin 2t$$

(b) $4y'' + 4y' + y = t^2 e^{-t/2} - 6t$

$$y_p = t^2 (At^2 + Bt + C) e^{-t/2} + Dt + E$$

3. Consider the Mass-Spring system

$$y'' + 9y = 9 \sin 3t. \quad (1)$$

(a) [16] Find a general solution to (1).

$$y_1 = \cos 3t, \quad y_2 = \sin 3t$$

$$y_p = At \cos 3t + Bt \sin 3t$$

$$y_p' = (A + 3Bt) \cos 3t + (B - 3At) \sin 3t$$

$$y_p'' = (6B - 9At) \cos 3t + (-6A - 9Bt) \sin 3t$$

$$y_p'' + 9y_p = (6B - 9At) \cos 3t + (-6A - 9Bt) \sin 3t + 9At \cos 3t + 9Bt \sin 3t$$

$$= 6B \cos 3t - 6A \sin 3t = 9 \sin 3t$$

$$\text{so } A = -\frac{3}{2}, \quad B = 0 \quad y_p = -\frac{3}{2}t \cos 3t \quad \boxed{y = C_1 \cos 3t + C_2 \sin 3t - \frac{3}{2}t \cos 3t}$$

Real Mucs
Since $9 \sin 3t = \text{Im}(9e^{3it})$ we can solve $z'' + 9z = 9e^{3it}$

$$z_p = At e^{3it}, \quad z_p' = (A + 3iAt) e^{3it}, \quad z_p'' = (6iA - 9At) e^{3it}$$

$$z_p'' + 9z_p = 6iA e^{3it} = 9e^{3it} \quad \text{so } A = \frac{9}{6i} = -\frac{3}{2}i \quad y_p = \text{Im}\left(-\frac{3}{2}i t e^{3it}\right) = -\frac{3}{2}t \cos 3t$$

$$W = 3, \quad y_p = \cos 3t \int \frac{-9 \sin 3t}{3} dt + \sin 3t \int \frac{9 \sin 3t \cos 3t}{3} dt$$

$$= -3 \cos 3t \cdot \frac{1}{2} (t - \frac{1}{6} \sin 6t) + \frac{1}{2} \sin^3 3t$$

$$\text{Note: } \frac{1}{6} \sin 6t = \frac{1}{3} \sin 3t \cos 3t$$

$$= -\frac{3}{2}t \cos 3t + \frac{1}{2} \sin 3t \cos^2 3t + \frac{1}{2} \sin^3 3t = -\frac{3}{2}t \cos 3t + \underbrace{\frac{1}{2} \sin 3t}$$

(b) [4] If the system (1) has initial data $y(0) = y'(0) = 0$, find the solution.

Part of homogeneous solution, not needed.

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y' = 3C_2 \cos 3t - \frac{3}{2}t \cos 3t + \frac{9}{2}t \sin 3t$$

$$y'(0) = 0 \Rightarrow 3C_2 - \frac{3}{2} = 0 \Rightarrow C_2 = \frac{1}{2}$$

$$y = \frac{1}{2} \sin 3t - \frac{3}{2}t \cos 3t$$

4. Reduction of Order.

- (a) [5] Use the Wronskian to verify that the second solution found via Reduction of Order is linearly independent from the first, i.e verify that if $y_1(t) \neq 0$, then $y_1(t)$ and

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) dt}}{(y_1(t))^2} dt$$

are linearly independent.

$$W = \begin{vmatrix} y_1 & y_1 \int e^{-\int p(t) dt} \\ y_1' & y_1' \int \frac{e^{-\int p(t) dt}}{y_1^2} + y_1 \cdot \frac{e^{-\int p(t) dt}}{y_1^2} \end{vmatrix} = e^{-\int p(t) dt} \neq 0 \text{ for all } t$$

- (b) [15] For $x > 0$, find a second linearly independent solution to

$$xy'' - y' + (1-x)y = 0$$

provided that $y_1 = e^x$ is a solution.

$$y_2 = e^x \int \frac{e^{-\int (-\frac{1}{x}) dx}}{e^{2x}} dx = e^x \int x e^{-2x} dx \quad u = x \quad du = dx \quad dv = e^{-2x} dx \quad v = -\frac{1}{2} e^{-2x}$$

$$= e^x \left[-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right]$$

$$= -\frac{x}{2} e^{-x} - \frac{1}{4} e^{-x}$$

5. Let $t > 0$ and $f(t)$ be continuous on $(0, \infty)$. Consider the equation

$$t^2 y'' - 4ty' + 4y = f(t). \quad (2)$$

(a) [5] Find a general solution to the homogeneous equation

$$t^2 y'' - 4ty' + 4y = 0.$$

$$r^2 - 5r + 4 = 0$$

$$(r - 4)(r - 1) = 0$$

$$y = C_1 t^4 + C_2 t$$

(b) [9] Find a particular solution to (2). Express your solution as a reasonably simplified combination of integrals, i.e. evaluate the Wronskian and combine like terms. **Do not** try to integrate an unknown function.

$$W \begin{bmatrix} t^4 & t \\ 4t^3 & 1 \end{bmatrix} = \begin{vmatrix} t^4 & t \\ 4t^3 & 1 \end{vmatrix} = t^4 - 4t^4 = -3t^4$$

$$y_p = t^4 \int \frac{-f(t) \cdot t}{t^2 \cdot (-3t^4)} dt + t \int \frac{f(t) \cdot t^4}{t^2 (-3t^4)} dt$$

$$= t^4 \int \frac{f(t)}{3t^5} dt - t \int \frac{f(t)}{3t^2} dt$$

6. [8] Find a solution of the form $y = C \cos(\omega t - \phi)$ that solves

$$y'' + 3y = 0, \quad y(0) = 1, y'(0) = 3.$$

$$y = C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t \quad y(0) = 1 \Rightarrow C_1 = 1$$

$$y' = -\sqrt{3} \sin \sqrt{3}t + \sqrt{3} C_2 \cos \sqrt{3}t \quad y'(0) = 3 \Rightarrow C_2 = \sqrt{3}$$

$$C = \sqrt{C_1^2 + C_2^2} = 2 \quad \tan \varphi = \sqrt{3} \quad \text{so } \varphi = \frac{\pi}{3} \quad (\text{since both } C_1, C_2 > 0)$$

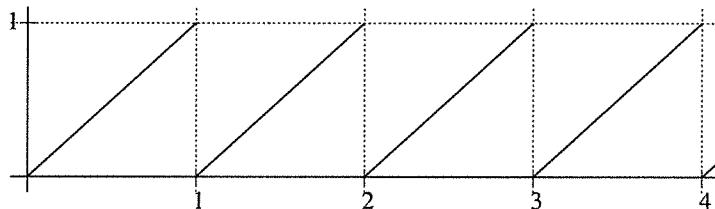
$$y = 2 \cos \left(\sqrt{3}t - \frac{\pi}{3} \right)$$

7. Laplace Transforms.

- (a) [4] Let $f(t)$ be defined for $t \in [0, \infty)$, state the integral definition of the Laplace transform of $f(t)$.

$$\mathcal{L} \{ f(t) \} = \int_0^\infty f(t) e^{-st} dt$$

- (b) [4] The sawtooth function defined by $f(t) = t$ for $t \in [0, 1)$ and $f(t+1) = f(t)$ for $t \geq 0$, i.e. $f(t)$ has period 1; see the figure below.



Use appropriate words to explain whether $f(t)$ has a Laplace transform. Do not compute the transform, if it exists.

f is piecewise continuous on $[0, \infty)$ and of exponential

order $\alpha = 0$, so f has a Laplace Transform.

8. [10] Using the provided table of Laplace transforms, show

$$\mathcal{L} \left\{ \frac{1}{2} (\sin 3t - 3t \cos 3t) \right\} = \frac{27}{(s^2 + 9)^2}.$$

$$\begin{aligned}
 \mathcal{L} \left\{ \frac{1}{2} (\sin 3t - 3t \cos 3t) \right\} &= \frac{1}{2} \left[\frac{3}{s^2 + 9} + 3 \frac{d}{ds} \left(\frac{s}{s^2 + 9} \right) \right] \\
 &= \frac{1}{2} \left[\frac{3}{s^2 + 9} + 3 \left(\frac{s^2 + 9 - 2s^2}{(s^2 + 9)^2} \right) \right] \\
 &= \frac{1}{2} \left[\frac{3s^2 + 27}{(s^2 + 9)^2} + \frac{27 - 3s^2}{(s^2 + 9)^2} \right] \\
 &= \frac{27}{(s^2 + 9)^2}
 \end{aligned}$$

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|--------|----|----|----|----|----|----|-------|
| Value | 20 | 20 | 20 | 14 | 16 | 10 | 100 |
| Points | | | | | | | |