

Exam 3 - Thursday, 5 April 2018

Show all appropriate work to receive full credit. Point values are in boxes

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at Montana State University, and that I have neither given nor received any unauthorized assistance (calculator, cell phone, notes, other people's exams, etc.) on this exam. Furthermore, I have and will not discuss the content of this exam, or provide any written or verbal aid to others. I understand that if I violate this statement, I will receive an exam score of 0."

SIGNATURE: _____

1. [10] Find the inverse Laplace transform of

$$F(s) = \frac{3s}{s^2 + 4s + 13}.$$

$$= \frac{3(s+2) - 6}{(s+2)^2 + 3^2}$$

$$f(t) = 3e^{-2t} \cos 3t - 2e^{-2t} \sin 3t$$

2. [15] Apply the Laplace transform to the initial value problem

$$y'' + 3y' - 4y = t \sin 2t, \quad y(0) = 1, y'(0) = -3$$

to express $Y(s) = \mathcal{L}\{y(t)\}$ in the form $Y(s) = \frac{P(s)}{Q(s)}$; express $P(s)$ as a combined simplified polynomial of the form $P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$ and $Q(s)$ as a fully factored polynomial.

Do not find the inverse Laplace transform.

$$s^2 Y - s + 3 + 3s Y - 3 - 4Y = - \frac{d}{ds} \left(\frac{2}{s^2 + 4} \right) = - \frac{-2(2s)}{(s^2 + 4)^2}$$

$$Y(s^2 + 3s - 4) = \frac{4s}{(s^2 + 4)^2} + s = \frac{4s + s^5 + 8s^3 + 16s}{(s^2 + 4)^2}$$

$$Y = \frac{s^5 + 8s^3 + 20s}{(s+4)(s-1)(s^2+4)^2}$$

3. [15] Applying the Laplace transform to the initial value problem

$$y'' + 9y = f(t), \quad y(0) = 0, y'(0) = 0$$

with

$$f(t) = \begin{cases} 9\sin 3t, & t < 2\pi \\ 18e^{-3t}, & t > 2\pi \end{cases}$$

gives

$$Y(s) = \frac{27}{(s^2 + 9)^2} + e^{-2\pi s} \left[\frac{18e^{-6\pi}}{(s+3)(s^2 + 9)} - \frac{27}{(s^2 + 9)^2} \right].$$

It will be useful to recall that on Exam 2 we showed that

$$\mathcal{L} \left\{ \frac{1}{2} (\sin 3t - 3t \cos 3t) \right\} = \frac{27}{(s^2 + 9)^2}.$$

You do not need to show this again.

Determine $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, the solution to the given initial value problem.

$$\frac{18}{(s+3)(s^2+9)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+9}$$

$$18 = A(s^2 + 9) + (Bs + C)(s + 3)$$

$$s = -3 : 18 = 18A \quad \text{so} \quad A = 1$$

$$s^2 : 0 = A + B \quad \text{so} \quad B = -1$$

$$s^0 : 18 = 9 + 3C \quad \text{so} \quad C = 3$$

$$y(t) = \frac{1}{2} (\sin 3t - 3t \cos 3t) + u(t-2\pi) \left[e^{-6\pi} \left\{ e^{-3(t-2\pi)} - \cos(3(t-2\pi)) + \sin(3(t-2\pi)) \right\} \right. \\ \left. - \frac{1}{2} \left(\sin(3(t-2\pi)) - 3(t-2\pi) \cos(3(t-2\pi)) \right) \right]$$

4. [15] Applying the substitution

$$x = y' + y \quad (1)$$

to the symbolic system of equations

$$\begin{aligned} x' &= x - 10y + 3\delta(t-1), & x(0) &= 1 \\ y' &= x - y, & y(0) &= 1 \end{aligned}$$

converts the system into the symbolic initial value problem

$$y'' + 9y = 3\delta(t-1), \quad y(0) = 1, y'(0) = 0. \quad (2)$$

- (a) Solve the initial value problem (2) for $y(t)$.

$$s^2 Y - s + 9Y = 3e^{-s}$$

$$Y(s^2 + 9) = 3e^{-s} + s$$

$$Y = \frac{3e^{-s}}{s^2 + 9} + \frac{s}{s^2 + 9}$$

$$y = u(t-1) \sin(3t-3) + \cos 3t$$

- (b) Using the substitution (1) and the fact that $\frac{d}{dt}u(t-a) = \delta(t-a)$, find $x(t)$.

$$y' = \delta(t-1) \sin(3t-3) + 3u(t-1) \cos(3t-3) - 3 \sin 3t$$

so

$$x = \delta(t-1) \sin(3t-3) + 3u(t-1) \cos(3t-3) - 3 \sin 3t$$

$$+ u(t-1) \sin(3t-3) + \cos 3t$$

5. Assume $g(t)$ is piecewise continuous and of exponential order and consider the initial value problem

$$y'' - 3y' + 2y = g(t), \quad y(0) = 1, y'(0) = 1.$$

- (a) [10] Find the solution. Express your solution in terms of a convolution.

$$s^2 Y - s - 1 - 3s Y + 3 + 2Y = G(s)$$

$$Y(s^2 - 3s + 2) = G(s) + s - 2$$

Aside:

$$Y = G(s) \left(\frac{1}{(s-1)(s-2)} \right) + \frac{s-2}{(s-1)(s-2)}$$

$$\frac{1}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$$

$$Y = G(s) \left[\frac{1}{s-2} - \frac{1}{s-1} \right] + \frac{1}{s-1}$$

$$1 = A(s-1) + B(s-2)$$

$$s=1 \Rightarrow B = -1$$

$$s=2 \Rightarrow A = 1$$

$$y = g(t) * (e^{2t} - e^t) + e^t$$

- (b) [5] Express the convolution in (a) in terms of the appropriate integral. Do not evaluate the integral.

$$g(t) * (e^{2t} - e^t) = \int_0^t g(t-v) (e^{2v} - e^v) dv$$

$$(e^{2t} - e^t) * g(t) = \int_0^t (e^{2(t-v)} - e^{t-v}) g(v) dv$$

6. [15] Consider the mass-spring system given by the initial value problem

$$x'' + 2x' + 5x = 0, \quad x(0) = 0, x'(0) = 2. \quad (3)$$

- (a) Find the solution to (3).

$$s^2 X - 2 + 2s X + 5X = 0$$

$$X(s^2 + 2s + 5) = 2$$

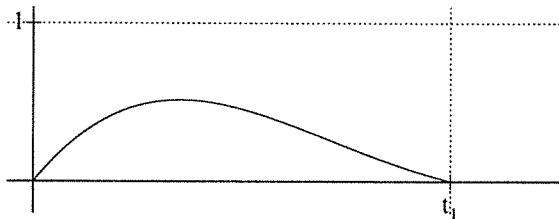
$$X = \frac{2}{(s+1)^2 + 2^2}$$

$$x = e^{-t} \sin 2t$$

- (b) Find the magnitude of the impulse needed to stop the motion of the system when it first returns to equilibrium at time $t_1 = \pi/2$, i.e. find M so that the solution to the symbolic initial value problem

$$x'' + 2x' + 5x = M\delta(t - \pi/2), \quad x(0) = 0, x'(0) = 2$$

has the following graph.



You may find the following useful.

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$X(s^2 + 2s + 5) = 2 + M e^{-\pi/2 s}$$

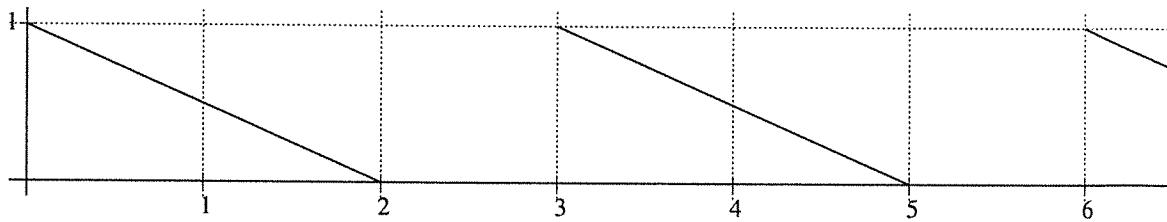
$$X = \frac{2}{(s+1)^2 + 2^2} + \frac{M}{2} e^{-\pi/2 s} \cdot \frac{2}{(s+1)^2 + 2^2}$$

$$\begin{aligned} x(t) &= e^{-t} \sin 2t + \frac{M}{2} u(t - \frac{\pi}{2}) e^{-(t-\frac{\pi}{2})} \sin(2t - \pi) \\ &= e^{-t} \sin 2t + \frac{M}{2} u(t - \frac{\pi}{2}) e^{-t} \cdot e^{\frac{\pi}{2}} \cdot (-\sin 2t) \end{aligned}$$

$$= e^{-t} \sin 2t \left[1 - \frac{M}{2} e^{\frac{\pi}{2}} u(t - \frac{\pi}{2}) \right]$$

$$\text{so } M = 2 e^{-\pi/2}$$

7. [15] Compute the Laplace transform of the periodic function $f(t)$ given by the graph below.



$$f_3(t) = \begin{cases} 1 - \frac{t}{2} & 0 < t < 2 \\ 0 & \text{otherwise} \end{cases} = 1 - \frac{t}{2} - u(t-2)(1 - \frac{t}{2})$$

$\leftarrow \quad 1 - \frac{t}{2} - 1 = -\frac{t}{2}$

$$F_3 = \frac{1}{s} - \frac{1}{2s^2} - e^{-2s} \int \left\{ 1 - \frac{t+2}{2} \right\}$$

$$= \frac{1}{s} - \frac{1}{2s^2} - e^{-2s} \left(-\frac{1}{2s^2} \right)$$

$$= \frac{1}{s} - \frac{1}{2s^2} + \frac{e^{-2s}}{2s^2}$$

$$F = \frac{\frac{1}{s} - \frac{1}{2s^2} + \frac{e^{-2s}}{2s^2}}{1 - e^{-3s}}$$

$$= \frac{2s - 1 + e^{-2s}}{2s^2(1 - e^{-3s})}$$

Page	1	2	3	4	5	6	Total
Value	25	15	15	15	15	15	100
Points							