4. $y'' + 5y' - 6y = e^{3t}$ $\quad y(0) = 2$, $y'(0) = 0$

$$s^2Y - 2s + 5(sY - 2) - 6Y = \frac{1}{s-3}$$

$$Y(s^2 + 5s - 6) = \frac{1}{s-3} + 2s + 10 = \frac{1 + (2s+10)(s-3)}{s-3}$$

$$Y = \frac{2s^2 + 4s - 29}{(s-3)(s+6)(s-1)}$$

5. A. $y' + 3y = g(t)$ $\quad y(0) = 2$

$$sY - 2 + 3Y = G(s)$$

$$Y(s+3) = G(s) + 2$$

$$Y = G(s) \cdot \frac{1}{s+3} + \frac{2}{s+3}$$

$$y = g(t) * e^{-3t} + 2e^{-3t}$$

B. $y = 3t * e^{-3t} + 2e^{-3t}$

$$= \int_0^t 3(t-v)e^{-3v}\,dv + 2e^{-3t}$$

$u = 3(t-v)$ $\quad dV = e^{-3v}dv$

$du = -3\,dv$ $\quad V = -\frac{1}{3}e^{-3v}$

$$= -(t-v)e^{-3v}\Big|_0^t - \int_0^t e^{-3v}dv + 2e^{-3t}$$

$$= t + \frac{1}{3}e^{-3v}\Big|_0^t + 2e^{-3t}$$

$$= t + \frac{1}{3}e^{-3t} - \frac{1}{3} + 2e^{-3t}$$

$$= \frac{7}{3}e^{-3t} + t - \frac{1}{3}$$

6. $y'' + 2\pi y' + 5\pi^2 y = 4\pi\,\delta(t-1)$ $\qquad y(0)=0\ ,\ y'(0)=2\pi$

$$s^2 Y - 2\pi + 2\pi s Y + 5\pi^2 Y = 4\pi e^{-s}$$

$$Y(s^2 + 2\pi s + 5\pi^2) = 4\pi e^{-s} + 2\pi$$

$$Y = \frac{4\pi}{(s+\pi)^2 + (2\pi)^2} e^{-s} + \frac{2\pi}{(s+\pi)^2 + (2\pi)^2}$$

$$y = 2u(t-1)\, e^{-\pi(t-1)} \sin(2\pi(t-1)) + e^{-\pi t} \sin(2\pi t)$$

$$= \begin{cases} e^{-\pi t} \sin(2\pi t) & t < 1 \\ (2e^{\pi}+1)\, e^{-\pi t} \sin(2\pi t) & t \geq 1 \end{cases}$$