

Math 274

Due: 20 Apr 2018

MUCs/VoP Thing

Show Appropriate Work

Name: _____

Point Values in .

1. We are interested in solving the nonhomogeneous equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \text{ and } \mathbf{f}(t) = \begin{bmatrix} 3 + 5e^{2t} \\ -6 + 2e^{2t} \end{bmatrix}.$$

We will do so by breaking it into the following pieces.

- (a) 1 Find a general solution to the homogeneous equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\begin{bmatrix} 1-r & 2 \\ 0 & 3-r \end{bmatrix} = (1-r)(3-r)$$

$$r=1 \quad \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{0} \quad \vec{x}_h = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$r=3 \quad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$$

- (b) 2 Use the Method of Undetermined Coefficients to find a particular solution to

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{bmatrix} 3 \\ -6 \end{bmatrix}.$$

Try $\vec{x}_p = \begin{bmatrix} a \\ b \end{bmatrix}$,

$$\Rightarrow \vec{A} \begin{bmatrix} -3 \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ b \end{bmatrix}$$

substituting gives

$$\vec{0} = \vec{A} \begin{bmatrix} 2 \\ b \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$$\frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ b \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ b \end{bmatrix} = \vec{x}_p$$

- (c) 2 Use the Method of Undetermined Coefficients to find a particular solution to

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{bmatrix} 5e^{2t} \\ 2e^{2t} \end{bmatrix}.$$

Try $\vec{x}_p = e^{2t} \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} 2a \\ 2b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

substituting gives

$$2a = 3b + 2 \quad \text{so } b = -2$$

$$2a = a + 2b + 5 \quad \text{so } a = 1$$

$$\text{so } \vec{x}_p = \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}$$

- (d) 1 Find a general solution to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$.

$$\vec{x} = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} e^{2t} - 7 \\ -2e^{2t} + 2 \end{bmatrix}$$

2. [4] Find a general solution for the nonhomogeneous equation $\mathbf{x}'(t) = \mathbf{Bx}(t) + \mathbf{g}(t)$ where

$$\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \text{ and } \mathbf{g}(t) = \begin{bmatrix} 18e^t \\ 9e^{4t} \end{bmatrix}.$$

The Method of Undetermined Coefficients is inconvenient in this case. Use Variation of Parameters to find a particular solution and then find a general solution.

$$\begin{vmatrix} 2-r & 1 \\ 2 & 3-r \end{vmatrix} = (2-r)(3-r) - 2 = r^2 - 5r + 4 = (r-1)(r-4)$$

$$r=1 : \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}$$

$$r=4 : \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{0}$$

$$\underline{\mathbf{X}} = \begin{bmatrix} e^t & e^{4t} \\ -e^t & 2e^{4t} \end{bmatrix} \text{ is a fundamental matrix}$$

$$\underline{\mathbf{X}}^{-1} = \frac{1}{3e^{5t}} \begin{bmatrix} 2e^{4t} & -e^{4t} \\ e^t & e^t \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t} & -e^{-t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$$

$$\underline{\mathbf{X}}^{-1} \underline{\mathbf{g}} = \frac{1}{3} \begin{bmatrix} 2e^{-t} & -e^{-t} \\ e^{-4t} & e^{-4t} \end{bmatrix} \begin{bmatrix} 18e^t \\ 9e^{4t} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 36 - 9e^{3t} \\ 18e^{-3t} + 9 \end{bmatrix} = \begin{bmatrix} 12 - 3e^{3t} \\ 6e^{-3t} + 3 \end{bmatrix}$$

$$\int \underline{\mathbf{X}}^{-1} \underline{\mathbf{g}} = \begin{bmatrix} 12t - e^{3t} \\ -2e^{-3t} + 3t \end{bmatrix}$$

$$\underline{\mathbf{x}}_p = \underline{\mathbf{X}} \int \underline{\mathbf{X}}^{-1} \underline{\mathbf{g}} = \begin{bmatrix} e^t & e^{4t} \\ -e^t & 2e^{4t} \end{bmatrix} \begin{bmatrix} 12t - e^{3t} \\ -2e^{-3t} + 3t \end{bmatrix} = \begin{bmatrix} 12te^t - e^{4t} - 2e^t + 3te^{4t} \\ -12te^t + e^{4t} - 4e^t + 6te^{4t} \end{bmatrix}$$

$$\underline{\mathbf{x}} = \begin{bmatrix} e^t & e^{4t} \\ -e^t & 2e^{4t} \end{bmatrix} \vec{C} + e^t \begin{bmatrix} 12t - 2 \\ -12t - 4 \end{bmatrix} + e^{4t} \begin{bmatrix} 3t - 1 \\ 6t + 1 \end{bmatrix} \text{ is a general solution.}$$