

Math 274
Due: 23 Apr 2018

Matrix Exponential
Show Appropriate Work

Name: _____
Point Values in boxes.

In Section 9.8 we saw that if $\mathbf{X}(t)$ is a Fundamental Matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ then

$$e^{\mathbf{At}} = \mathbf{X}(t)\mathbf{X}^{-1}(0). \quad (1)$$

For any 2×2 matrix \mathbf{A} , the matrix exponential $e^{\mathbf{At}}$ can be computed according to the table below.

Eigenvalues of \mathbf{A}	$e^{\mathbf{At}}$
r_1, r_2 real and distinct	$e^{r_1 t} \frac{1}{r_1 - r_2} (\mathbf{A} - r_2 \mathbf{I}) - e^{r_2 t} \frac{1}{r_2 - r_1} (\mathbf{A} - r_1 \mathbf{I})$
r real repeated twice	$e^{rt} \mathbf{I} + t e^{rt} (\mathbf{A} - r \mathbf{I})$
$\alpha \pm i\beta$ complex conjugate pair	$e^{\alpha t} \cos(\beta t) \mathbf{I} + e^{\alpha t} \sin(\beta t) \frac{1}{\beta} (\mathbf{A} - \alpha \mathbf{I})$

1. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$.

- (a) Compute $e^{\mathbf{At}}$ using (1) above.

$$r_1 = 2, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, r_2 = 1, \vec{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ so } \underline{\mathbf{X}} = \begin{bmatrix} e^{2t} & 3e^t \\ e^{2t} & 4e^t \end{bmatrix}$$

$$\underline{\mathbf{X}}(0) = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \underline{\mathbf{X}}^{-1}(0) = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\underline{\mathbf{X}} \underline{\mathbf{X}}^{-1}(0) = \begin{bmatrix} 4e^{2t} - 3e^t & 3e^t - 3e^{2t} \\ 4e^{2t} - 4e^t & 4e^t - 3e^{2t} \end{bmatrix}$$

- (b) Compute $e^{\mathbf{At}}$ using the formula.

$$e^{2t} \left(\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) - e^t \left(\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} 4e^{2t} - 3e^t & 3e^t - 3e^{2t} \\ 4e^{2t} - 4e^t & 4e^t - 3e^{2t} \end{bmatrix}$$

- (c) Note, $e^{\mathbf{At}}$ is unique, so your solutions should be the same.

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2. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$.

(a) Compute $e^{\mathbf{At}}$ using (1).

$$r = -1 \pm 2i, \quad \vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{\mathbf{X}} = \begin{bmatrix} -e^{-t} \cos 2t & -e^{-t} \sin 2t \\ e^{-t} (\cos 2t - \sin 2t) & e^{-t} (\sin 2t + \cos 2t) \end{bmatrix}$$

$$\underline{\mathbf{X}}(0) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\underline{\mathbf{X}} \underline{\mathbf{X}}(0)^{-1} = e^{-t} \begin{bmatrix} -\cos 2t & -\sin 2t \\ \cos 2t - \sin 2t & \sin 2t + \cos 2t \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 2t - \sin 2t & -\sin 2t \\ 2 \sin 2t & \sin 2t + \cos 2t \end{bmatrix}$$

(b) Compute $e^{\mathbf{At}}$ using the formula.

$$e^{\mathbf{At}} = e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} e^{-t} \sin 2t \begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 2t - \sin 2t & -\sin 2t \\ 2 \sin 2t & \cos 2t + \sin 2t \end{bmatrix}$$

(c) Solve the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\underline{\mathbf{x}} = e^{\mathbf{At}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} 3 \cos 2t - 4 \sin 2t \\ \cos 2t + 7 \sin 2t \end{bmatrix}$$

3. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$.

(a) In Section 9.5 #35 we saw that \mathbf{A} had a repeated eigenvalue and found a solution making use of the idea of a *generalized eigenvector*. This was a non-trivial exercise.

(b) Compute $e^{\mathbf{At}}$ using the formula.

$$r = -1$$

$$e^{-t} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix}$$