1. Assume $g(t)$ is piecewise continuous and of exponential order and consider the initial value problem

$$y'' + 2y' + y = g(t), \quad y(0) = 1, y'(0) = 2.$$ 

(a) [4] Find the solution. Express your solution in terms of a convolution.

$$s^2 Y - s - 2 + 2s Y - 2 + Y = G(s)$$

$$Y(s^2 + 2s + 1) = G(s) + s + 4$$

$$Y = G\left(\frac{1}{(s+1)^2}\right) + \frac{s + 4}{(s+1)^2}$$

Note: $$\frac{5+4}{(5+1)^2} = \frac{5+1}{(5+1)^2} - \frac{3}{(5+1)^2}$$

$$y = g(t) * te^{-t} + e^{-t} + 3te^{-t}$$

(b) [1] Express the convolution in part (a) as an appropriate integral.

$$g(t) * te^{-t} = \int_{0}^{t} g(t-v)ve^{-v}dv$$

CONTINUED ON REVERSE.
2. Consider a mass-spring system sitting in front of a cuckoo clock. After \( \pi \) seconds the time is exactly 1 pm. The cuckoo comes out of the clock and strikes the system exerting an impulse on the mass. The system is governed by the symbolic initial value problem

\[
x'' + 4x = 2\delta(t - \pi), \quad x(0) = 0, x'(0) = -2,
\]

(1)

where \( x(t) \) measures the displacement from the equilibrium.

(a) Determine \( x(t) \), i.e. solve the symbolic initial value problem (1).

\[
s^2 X(s) + 2sX(s) + 4X(s) = 2e^{-\pi s}
\]

\[
X(s) = \frac{-2}{s^2 + 4} + \frac{2}{s^2 + 4} e^{-\pi s}
\]

\[
x(t) = -\sin 2t + u(t - \pi) \sin \left( \frac{2(t - \pi)}{2} \right) - \sin (2t - 2\pi) - \sin 2t
\]

\[
x(t) = \begin{cases} 
-\sin 2t & 0 < t < \pi \\
0 & \text{otherwise}
\end{cases}
\]

(b) Carefully sketch a graph of \( x(t) \) for \( t \in [0, 2\pi] \).