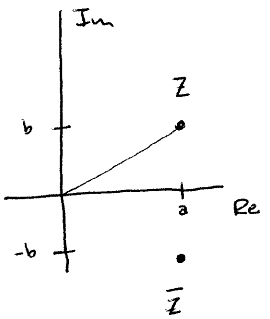
Complex Numbers, an introduction

* There is a really good article on the NYTimes, linked to on my web page. *

Define $i := \sqrt{-1}$ or $i^2 = -1$

A complex number is of the form $z = a + ib$, $a, b \in \mathbb{R}$, $z \in \mathbb{C}$



The ~~magnitude~~ modulus is

$$|z| = \sqrt{a^2 + b^2}$$ (or magnitude)

The conjugate is

$$\bar{z} = a - ib$$

The real part is $\operatorname{Re}(z) = a \in \mathbb{R}$

The imaginary —"— $\operatorname{Im}(z) = b \in \mathbb{R}$

Complex numbers are equal iff the real parts are equal & the imaginary parts are equal.

Basic Operations

Let $z = a + ib$, $w = c + id$

$$z + w = (a + ib) + (c + id) = (a + c) + i(b + d)$$

$$zw = (a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

so $$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$$

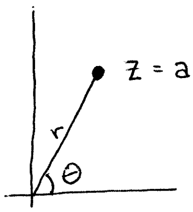
so $|z| = \sqrt{z\bar{z}}$

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}$$

Example

$$\frac{2+i}{3-i} = \frac{2+i}{3-i} \cdot \frac{3+i}{3+i} = \frac{5+5i}{9+1} = \frac{1}{2} + \frac{1}{2} i$$ ← This is important!

Polar Representation



$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Enter: $e^{i\theta} := \cos\theta + i\sin\theta$ defn/formula

Does it make sense to use exponential language?

1) Does it satisfy exponential laws, in particular $a^x a^y = a^{x+y}$?

2) $y = e^{at}$ solves $y' = ay$, $y(0) = 1$. Does $Y = e^{i\theta}$ solve $Y' = iY$, $Y(0) = 1$?

In order,

1) $e^{i\theta} e^{i\varphi} \stackrel{?}{=} e^{i(\theta+\varphi)}$

$$
\begin{aligned}
e^{i\theta} e^{i\varphi} &= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\
&= \cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi) \\
&= \cos(\theta+\varphi) + i(\sin(\theta+\varphi)) = e^{i(\theta+\varphi)} \quad ✓
\end{aligned}
$$

2) $\frac{d}{d\theta} e^{i\theta} \stackrel{?}{=} ie^{i\theta}$, $e^{i(0)} = 1$

To differentiate a complex valued function $f: \mathbb{R} \to \mathbb{C}$, $f(x) = u(x) + iv(x)$

$u(x), v(x)$ real valued.

$f'(x) = u'(x) + iv'(x)$

so

$$\frac{d}{d\theta} e^{i\theta} = \frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i\left(\cos\theta - \frac{1}{i}\sin\theta\right) = i(\cos\theta + i\sin\theta) \stackrel{✓}{=} ie^{i\theta}$$

$\frac{1}{i} = \frac{i}{-1} = -i$

$e^{i(0)} = \cos(0) + i\sin(0) \stackrel{✓}{=} 1$

3) Series ?

One additional question,

3) Assuming that power series work in $\mathbb{C}$ as well as $\mathbb{R}$, is the definition consistent with what we know from 172?

From 172 we have

$$\forall_{x \in \mathbb{R}} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

so

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\frac{\theta}{1!} + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots$$

Since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ⋮

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta \quad ✓$$

More on Polar Form

$z = re^{i\theta}$ $\qquad r = $ modulus of z , $|z|$

$\theta = $ arguement of z , $\arg(z)$

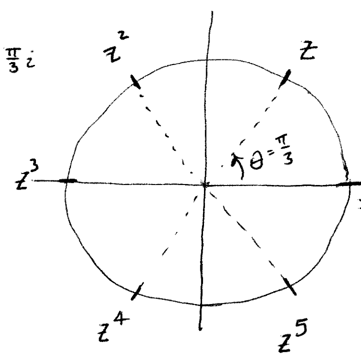
Multiplication in Polar

$$r_1 e^{i\theta_1} \, r_2 e^{i\theta_2} = \underbrace{r_1 r_2}_{\text{multiply modulus}} e^{i\overbrace{(\theta_1+\theta_2)}^{\text{Add arguement}}}$$

so powers

$$(re^{i\theta})^n = r^n e^{in\theta}$$

so if $z = e^{\frac{\pi}{3}i}$



$z^6 = (e^{\pi/3 i})^6 = e^{2\pi i} = 1$

Note:

$z = e^{\pi/3 i}$ is a 6^{th} root of unity, i.e. $(e^{\pi/3 i})^6 = 1$

Find the 5^{th} roots of unity, i.e. solve $z^5 = 1$

$(re^{i\theta})^5 = 1$ $\quad$ so $r = 1$, $\quad e^{5\theta i} = e^{i0} = e^{i2\pi} = e^{i4\pi} = e^{i6\pi} = e^{i8\pi}$

so $5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\theta = 0, \frac{2}{5}\pi, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

In general

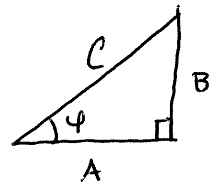
the n^{th} roots of unity are given by

$$z_k = e^{2\pi i k/n} \qquad k = 0, 1, \ldots, (n-1)$$

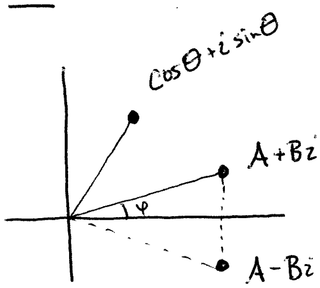
Application (i.e. Making things simplier by complexifying)

<u>Claim</u>: $A\cos\theta + B\sin\theta = C\cos(\theta - \varphi)$

where $C = \sqrt{A^2+B^2}$, $\tan\varphi = \frac{B}{A}$



<u>Pf</u>:



$$(\cos\theta + i\sin\theta)(A - iB)$$

multiplying gives $A\cos\theta + B\sin\theta + i\,(A\sin\theta - B\cos\theta)$

since $\cos\theta + i\sin\theta = e^{i\theta}$

& $A - iB = \sqrt{A^2+B^2}\; e^{(-\varphi)i}$

then $e^{i\theta} \cdot e^{i(-\varphi)} = e^{i(\theta-\varphi)}$

so $(\cos\theta + i\sin\theta)(A - iB) = \sqrt{A^2+B^2}\,\big(\cos(\theta-\varphi) + i\sin(\theta-\varphi)\big)$

Equating real parts gives the desired result.

Application to Calculus

Complexify the integral. $\cos x = Re(e^{ix})$

$$\int e^{-x}\cos x\,dx = Re\int e^{-x}e^{ix}\,dx = Re\int e^{(-1+i)x}\,dx$$

in 172 we did parts twice

Ignore the $+C$

(the C in this integral is complex)

$$= Re\left(\frac{1}{-1+i}e^{(-1+i)x}\right)$$

$$= Re\left(\frac{-1-i}{1+1}e^{-x}e^{ix}\right)$$

$$= Re\left(\frac{-1-i}{2}e^{-x}\left(\cos x + i\sin x\right)\right)$$

$$= Re\left(\frac{e^{-x}}{2}\left[-\cos x + \sin x + i\left(-\cos x - \sin x\right)\right]\right)$$

$$= \frac{1}{2}e^{-x}\left(\sin x - \cos x\right) + C$$

this is a real valued constant.