Math 274 — 06 July 2017

Exam 1 — Show Appropriate Work

Name: ____________

Point Values $\boxed{\text{boxes}}$.

Express your solutions as explicit functions unless otherwise specified.

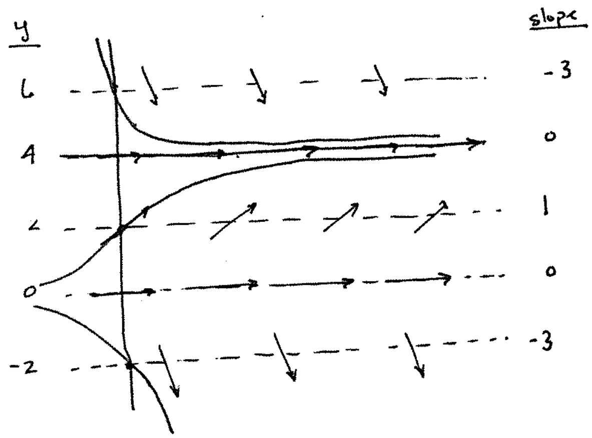
1. Consider the autonomous differential equation $\dfrac{dp}{dt} = \dfrac{p(4-p)}{4}$.

   (a) $\boxed{5}$ Sketch the isoclines passing through $(0,-2)$, $(0,0)$, $(0,2)$, $(0,4)$, and $(0,6)$. Include direction arrows.

   (b) $\boxed{4}$ Sketch the solution curves passing through $(0,-2)$, $(0,2)$, $(0,4)$, and $(0,6)$.

   (c) $\boxed{10}$ Find the general solution. Did you lose any solutions?

   (d) $\boxed{1}$ Solve the IVP $p(0)=2$.



$$\frac{dp}{dt} = p\frac{(4-p)}{4} \quad \text{so} \quad \int \frac{4}{p(4-p)}\,dp = \int dt$$

Note: $\dfrac{4}{p(4-p)} = \dfrac{A}{p} + \dfrac{B}{4-p}$

$4 = A(4-p) + Bp$

$p=0 \Rightarrow A=1$

$p=4 \Rightarrow B=1$

so $\displaystyle\int \left(\frac{1}{p} + \frac{1}{4-p}\right)dp = \int dt$

or $\ln|p| - \ln|4-p| = t + C$

$\dfrac{p}{4-p} = Ce^t$

$p = 4Ce^t - Ce^t p$

Finally, $p = \dfrac{4Ce^t}{1+Ce^t}$

$p \equiv 0$ & $p \equiv 4$ were lost, but the first is reclaimed by letting $C=0$, so only $p\equiv 4$ remains lost.

The solution to the IVP is

$$p = \frac{4e^t}{1+e^t} \quad (\text{i.e. } C=1).$$

2. $\boxed{15}$ Solve the initial value problem

$$\frac{dy}{dt} = 4t + 2ty, \qquad y(0) = 1.$$

$$\frac{dy}{dt} - 2t\,y = 4t \quad \text{so} \quad \mu(t) = e^{-t^2}.$$

$$y = e^{t^2}\int 4te^{-t^2}\,dt = e^{t^2}\left[-2e^{-t^2} + c\right] = Ce^{t^2} - 2$$

is the general solution.

The solution to the IVP is

$$y = 3e^{t^2} - 2.$$

Note: This equation is also separable.

$$\int \frac{dy}{2+y} = \int 2t\,dt$$

$$\ln|2+y| = t^2 + c$$

$$2+y = Ce^{t^2}$$

$$y = Ce^{t^2} - 2.$$

3. Consider the equation $(1 - y\sin(xy))dx + (\cos y - x\sin(xy))dy = 0$.

(a) $\boxed{3}$ Show the equation is exact.

$$\frac{\partial M}{\partial y} = -\sin(xy) - xy\cos(xy)$$

$$\frac{\partial N}{\partial x} = -\sin(xy) - xy\cos(xy)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact!

(b) $\boxed{12}$ Find the general solution. An implicit solution is expected.

$$x + \cos xy + \sin y = C$$

4. Using an appropriate substitution, find the general solution for the following.

(a) $\boxed{15}$ $\quad \dfrac{dy}{dx} = \dfrac{2y^2 - x^2}{xy} = \dfrac{2\left(\frac{y}{x}\right)^2 - 1}{\left(\frac{y}{x}\right)}$ $\quad$ Let $v = \frac{y}{x}$ so $xv = y$ so $v + x\frac{dv}{dx} = \frac{dy}{dx}$

Homogeneous

Substituting gives

$v + x\dfrac{dv}{dx} = \dfrac{2v^2-1}{v}$ so $x\dfrac{dv}{dx} = \dfrac{v^2-1}{v}$. Now separate $\displaystyle\int \frac{v}{v^2-1}\,dv = \int \frac{dx}{x}$. Integrating yields

$\frac{1}{2}\ln|v^2-1| = \ln|x| + C$ so $v^2 = Cx^2+1$ & finally $y = \pm\sqrt{Cx^4+x^2}$.

Bernoulli

Although it wasn't inteded to be, this equation is also Bernoulli.

$\dfrac{dy}{dx} - \dfrac{2y}{x} = -\dfrac{x}{y}$ so $y\dfrac{dy}{dx} - \dfrac{2y^2}{x} = -x$. Let $v = y^2$ so $\dfrac{dv}{dx} = 2y\dfrac{dy}{dx}$.

Substituting & Simplifying gives

$\dfrac{dv}{dx} - \dfrac{4v}{x} = -2x$ so $\mu(x) = x^{-4}$ & $v = x^4\displaystyle\int -2x^{-3}\,dx = x^4\left[x^{-2}+c\right]$.

As above we have $y = \pm\sqrt{x^2+Cx^4}$.

(b) $\boxed{15}$ $\quad \dfrac{dx}{dt} = x + \dfrac{t}{x^2}$ is Bernoulli; $\dfrac{dx}{dt} - x = \dfrac{t}{x^2}$ so $x^2\dfrac{dx}{dt} - x^3 = t$

Let $v = x^3$ so $3x^2\dfrac{dx}{dt} = \dfrac{dv}{dt}$. Substituting and Simplifying gives

$\dfrac{dv}{dt} - 3v = 3t$ so $\mu(t) = e^{-3t}$ & $v = e^{3t}\displaystyle\int 3te^{-3t}\,dt$.

Applying Integration by Parts to the integral yields

$$v = e^{3t}\left[-te^{-3t} - \tfrac{1}{3}e^{-3t} + C\right] = Ce^{3t} - t - \tfrac{1}{3}.$$

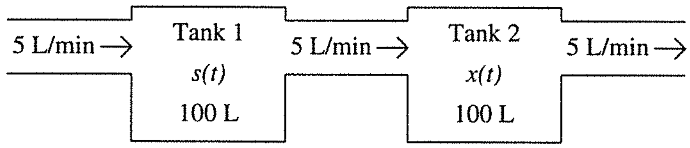
An explicit solution is then

$$x = \left(Ce^{3t} - t - \tfrac{1}{3}\right)^{1/3}$$

HINT: In one of the above you likely lost valid solutions, make sure you find them.

Note: In the Homogeneous solution to (a) solutions of the form $v = \pm 1$ were lost, i.e. $y = \pm x$. However they are regained when $C = 0$.

5. Consider a two tank system as in the figure.



(a) The first tank is initially filled with 100 L of pure water. A brine solution containing 0.2 kg of salt per liter is being pumped into the tank at a rate of 5 L/min. The tank is well mixed and drains at 5 L/min. Let s be the amount of salt (in kg) in the tank at time t (in min).

i. $\boxed{4}$ Set up an initial value problem (a differential equation with an initial condition) modeling the amount of salt in the tank at time t. **DO NOT SOLVE.**

$$\frac{ds}{dt} = 5(0.2) - 5\left(\frac{s}{100}\right) \qquad s(0) = 0$$

ii. $\boxed{2}$ As $t \to \infty$, what can you say about the amount of salt in the tank?

$$s \longrightarrow 20 \text{ Kg} \quad \text{as} \quad t \longrightarrow \infty$$

(b) A second tank is initially filled with 100 L of a brine solution with a concentration of 0.3 kg/L. The solution flowing out of the first tank flows into the second tank at 5 L/min. The second tank is also well stirred and drains at 5 L/min. Let x be the amount of salt (in kg) in the tank at time t (in min).

i. $\boxed{12}$ Determine the amount of salt in the second tank at any time t. Use the fact that the amount of salt in the first tank (in kg) is given by $s(t) = 20 - 20e^{-t/20}$.

$$\frac{dx}{dt} = 5\left(\frac{20-20e^{-t/20}}{100}\right) - \frac{5x}{100} \qquad x(0) = 30$$

$$\frac{dx}{dt} = 1 - e^{-t/20} - \frac{5x}{100}$$

$$\frac{dx}{dt} + \frac{5x}{100} = 1 - e^{-t/20} \qquad \mu(t) = e^{5t/100} = e^{t/20}$$

$$x = e^{-t/20}\int (e^{t/20} - 1)\,dt = e^{-t/20}\left[20e^{t/20} - t + C\right] = 20 + e^{-t/20}(C - t)$$

Since $x(0) = 30$ $\quad C = 10$ & we have

$$x(t) = 20 + e^{-t/20}(10 - t)$$

ii. $\boxed{2}$ When is the concentration of salt in the second tank the highest?

$t = 0$