1. Consider the autonomous differential equation \( \frac{dp}{dt} = \frac{p(4-p)}{4} \).

(a) **5** Sketch the isoclines passing through \((0, -2), (0, 0), (0, 2), (0, 4),\) and \((0, 6)\). Include direction arrows.

(b) **4** Sketch the solution curves passing through \((0, -2), (0, 2), (0, 4),\) and \((0, 6)\).

(c) **10** Find the general solution. Did you lose any solutions?

(d) **1** Solve the IVP \( p(0) = 2 \).

\[
\frac{dp}{dt} = \frac{p(4-p)}{4} \quad \text{so} \quad \int \frac{4}{p(4-p)} \, dp = \int dt
\]

Note: \( \frac{A}{p(4-p)}, \frac{A}{p}, \frac{B}{4-p} \)

\[
A = A(4-p) + Bp \quad \text{so} \quad \int \left( \frac{1}{p} + \frac{1}{4-p} \right) \, dp = \int dt \quad \text{or} \quad \ln |p| - \ln |4-p| = t + C
\]

\[
p = C e^t \quad \text{or} \quad \frac{p}{4-p} = C e^t \quad \text{or} \quad \frac{p}{4} = C e^t - C e^t p
\]

Finally, \( p = \frac{4 C e^t}{1 + C e^t} \)

\[
p = 0 \quad \text{or} \quad p = 4 \quad \text{were lost, but the root is reclaimed by letting } C = 0, \text{ so only } p = 4 \text{ remains lost.}
\]

The solution to the IVP is

\[
p = \frac{4 e^t}{1 + C e^t} \quad (\text{i.e. } C = 1).
\]
2. Solve the initial value problem

\[ \frac{dy}{dt} = 4t + 2ty, \quad y(0) = 1. \]

\[ \frac{dy}{dt} - 2ty = 4t \quad \text{so} \quad e^{-t} y = e^{-t} \int 4t e^t dt = e^{-t} \left[ -2e^{-t} + C \right] = Ce^t - 2 \]

is the general solution.

The solution to the IVP is

\[ y = 3e^t - 2. \]

Note: This equation is also separable.

\[ \int \frac{dy}{2y} = \int 2t dt \]

\[ \ln 2y = t^2 + C \]

\[ 2y = Ce^{t^2} \]

\[ y = Ce^{t^2} - 2. \]

3. Consider the equation \((1 - y \sin(xy))dx + (\cos y - x \sin(xy))dy = 0.\)

(a) Show the equation is exact.

\[ \frac{\partial M}{\partial y} = -\sin(xy) - xy \cos(xy) \]

Since \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \), the equation is exact!

(b) Find the general solution. An implicit solution is expected.

\[ \chi + \cos \chi \cdot y + \sin y = C \]
4. Using an appropriate substitution, find the general solution for the following.

(a) \[
\frac{dy}{dx} = \frac{2y^2 - x^2}{xy} = \frac{2\left(\frac{y}{x}\right)^2 - 1}{\left(\frac{y}{x}\right)}
\]
Let \( v = \frac{y}{x} \) so \( v = y \) so \( v + x \frac{dv}{dx} = \frac{dy}{dx} \)
Substituting gives
\[
\frac{v}{x} + \frac{dv}{dx} = \frac{2v^2 - 1}{v}
\]
so \( x\frac{dv}{dx} = \frac{v^2 - 1}{v} \). Now separate \( \frac{v}{\sqrt{v^2 - 1}} \) \( dv = \frac{dy}{x} \). Integrating \( v \) yields
\[
\frac{1}{2} \ln |v^2 - 1| = \ln |x| + C
\]
so \( v^2 = Cx^2 + 1 \) and finally \( y = \pm \sqrt{Cx^2 + x^2} \).

Although it wasn't intended to be, this equation is also Bernoulli.

(b) \[
\frac{dx}{dt} = x + \frac{t}{x^2}
\]
so \( \frac{dx}{dt} = \frac{t}{x^2} \). Letting \( v = \frac{y}{x} \) so \( \frac{dv}{dt} = \frac{y}{x} \). Substituting and simplifying gives
\[
\frac{dv}{dt} - 3v = 3t
\]
so \( \mu(t) = e^{-3t} \). Integrating \( \mu(t) \) yields
\[
v = e^{3t} \int 3te^{-3t} dt
\]
Applying Integration by Parts to the integral yields
\[
v = e^{3t} \left[ -te^{-3t} - \frac{1}{3} e^{-3t} + C \right] = Ce^{3t} - t - \frac{1}{3}.
\]
An explicit solution is then
\[
\kappa = \left( Ce^{3t} - t - \frac{1}{3} \right)^{\frac{1}{3}}
\]

HINT: In one of the above you likely lost valid solutions, make sure you find them.

Note: In the homogeneous solution to (a), solutions of the form \( v = \pm 1 \) were lost, i.e. \( y = \pm x \). However they are required when \( C = 0 \).
5. Consider a two tank system as in the figure.

\[ \begin{array}{c}
5 \text{ L/min} \rightarrow \\
\text{Tank 1} \\
\text{s(t)} \\
100 \text{ L} \\
5 \text{ L/min} \rightarrow \\
\text{Tank 2} \\
\text{x(t)} \\
100 \text{ L}
\end{array} \]

(a) The first tank is initially filled with 100 L of pure water. A brine solution containing 0.2 kg of salt per liter is being pumped into the tank at a rate of 5 L/min. The tank is well mixed and drains at 5 L/min. Let s be the amount of salt (in kg) in the tank at time t (in min).

i. [4] Set up an initial value problem (a differential equation with an initial condition) modeling the amount of salt in the tank at time t. **DO NOT SOLVE.**

\[ \frac{ds}{dt} = 5 \left( 0.2 \right) - 5 \left( \frac{s}{100} \right) \quad s(0) = 0 \]

ii. [2] As \( t \to \infty \), what can you say about the amount of salt in the tank?

\[ s \rightarrow 20 \]

(b) A second tank is initially filled with 100 L of a brine solution with a concentration of 0.3 kg/L. The solution flowing out of the first tank flows into the second tank at 5 L/min. The second tank is also well stirred and drains at 5 L/min. Let x be the amount of salt (in kg) in the tank at time t (in min).

i. [12] Determine the amount of salt in the second tank at any time \( t \). Use the fact that the amount of salt in the first tank (in kg) is given by \( s(t) = 20 - 20e^{-t/20} \).

\[ \frac{dx}{dt} = 5 \left( \frac{20 - 20e^{-t/20}}{100} \right) - \frac{5x}{100} \quad x(0) = 30 \]

\[ \frac{dx}{dt} = 1 - e^{-t/30} - \frac{5x}{100} \]

\[ \frac{dx}{dt} + \frac{5x}{100} = 1 - e^{-t/30} \quad x(t) = e^{5t/100} + e^{-t/30} \]

\[ x = e^{-t/30} \int \left( e^{t/30} - 1 \right) dt = e^{-t/30} \left[ 20e^{t/30} - t + C \right] = 20 + e^{-t/30} \left( C - t \right) \]

Since \( x(0) = 30 \), \( C = 10 \). If we have

\[ x(t) = 20 + e^{-t/30} \left( 10 - t \right) \]

ii. [2] When is the concentration of salt in the second tank the highest?