Math 274 | Exam 2 | Name: ____________
20 July 2017 | Show Appropriate Work | Point Values in $\boxed{\text{boxes}}$.

1. Find the general solution for the following.

   (a) $\boxed{4}$ $\quad 2y'' + 3y' - 2y = 0$

   $2r^2 + 3r - 2 = 0$

   $(2r - 1)(r + 2) = 0$

   $r = 1/2$, $\quad r = -2$

   $y = C_1 e^{t/2} + C_2 e^{-2t}$

   (b) $\boxed{4}$ $\quad y'' - 4y' + 13y = 0$

   $r = \dfrac{4 \pm \sqrt{16 - 52}}{2}$

   $= \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

   $y = C_1 e^{2t} \cos 3t + C_2 e^{2t} \sin 3t$

2. $\boxed{8}$ Solve the initial value problem.

$$y'' - y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 7$$

$r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$y = C_1 e^{3t} + C_2 e^{-2t}$ $\qquad y(0) = C_1 + C_2 = 4$

$y' = 3C_1 e^{3t} - 2C_2 e^{-2t}$ $\qquad y'(0) = 3C_1 - 2C_2 = 7$

so $\quad 2C_1 + 2C_2 = 8$

$3C_1 - 2C_2 = 7$

$5C_1 = 15 \Rightarrow C_1 = 3, \quad C_2 = 1$

$y = 3e^{3t} + e^{-2t}$

3. Find the appropriate form using the Method of Undetermined Coefficients for a particular solution to the following. **Do not** solve for the unknown constants.

(a) $\boxed{5}$ $\quad y'' - y = (t^2 - 1)e^t + 7$

$r^2 - 1 = 0$

$r = \pm 1$

$$y_p = t\left(At^2 + Bt + C\right)e^t + D$$

(b) $\boxed{5}$ $\quad y'' + 4y = 7t\cos 2t$

$r^2 + 4 = 0$

$r = \pm 2i$

$$y_p = t\left(At + B\right)\cos 2t + t\left(Ct + D\right)\sin 2t$$

4. $\boxed{10}$ Find the general solution to

$$y'' - 2y' + y = 4(2t + 1)e^{3t}.$$

$(r-1)^2 = 0$

so

$y_1 = e^t$ & $y_2 = te^t$ solve $y'' - 2y' + y = 0$.

Now, use MUC,

$$y_p = (At + B)e^{3t}$$

$$y_p' = [A + 3At + 3B]e^{3t}$$

$$y_p'' = [3A + 3A + 9At + 9B]e^{3t}$$

so

$$y_p'' - 2y_p' + y_p = [(9A - 6A + A)t + (6A + 9B - 2A - 6B + B)]e^{3t}$$

$$= [(4A)t + (4A + 4B)]e^{3t} = 4[At + (A+B)]e^{3t}$$

so $A = 2$, $B = -1$

$$y = C_1 e^t + C_2 te^t + (2t-1)e^{3t}$$

Let $t > 0$, for Cauchy-Euler equations of the form

$$at^2y'' + bty' + cy = 0 \tag{1}$$

we found that solutions were of the form $y = t^r$ where r is a root of the characteristic equation

$$ar^2 + (b-a)r + c = 0. \tag{2}$$

In the same section we discussed the Reduction of Order theorem. Specifically, let y_1 be a non-trivial solution, i.e. $y_1 \not\equiv 0$, to

$$y'' + p(t)y' + q(t)y = 0 \tag{3}$$

on an interval I. Then a second linearly independent solution to (3) is given by

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)\,dt}}{y_1(t)^2}\,dt. \tag{4}$$

5. $\boxed{10}$ Use Reduction of Order to show that if r is a double root of (2), and hence $y_1 = t^r$ is a solution to (1), then a second linearly independent solution to (1) is given by $y_2 = t^r \ln t$.

HINT 1: If r is a double root of (2) what can you say about $(b-a)^2 - 4ac$?

$$(b-a)^2 - 4ac = 0$$

HINT 2: With HINT 1 in mind, what can you say about r in terms of a and b?

$$\text{so} \quad r = \frac{-(b-a)}{2a} = \frac{a-b}{2a}$$

HINT 3: Before you dive into the formula in (4) make sure you are using the form given in (3).

$$y'' + \frac{b}{at}y' + \frac{c}{at^2} = 0 \qquad \text{so} \quad e^{-\int p} = e^{-\int \frac{b}{at}\,dt} = e^{-\frac{b}{a}\ln t} = t^{-b/a}$$

$$y_2 = t^r \int \frac{t^{-b/a}}{t^{2r}}\,dt \qquad \text{but} \quad r = \frac{a-b}{2a} \quad \text{so} \quad t^{2r} = t^{\frac{a-b}{a}} = t^{1-\frac{b}{a}}$$

$$= t^r \int \frac{t^{-b/a}}{t^{1-b/a}}\,dt = t^r \int \frac{dt}{t} = t^r \ln t$$

$\blacksquare$

6. Consider the equation $t^2y'' - 4ty' + 4y = 3t^4 \ln t$.

(a) $\boxed{4}$ Find the general solution for

$$t^2y'' - 4ty' + 4y = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

so $y = C_1 t + C_2 t^4$.

(b) $\boxed{14}$ Use Variation of Parameters with

$$v_1 = \int \frac{-g(t)y_2(t)}{W[y_1, y_2](t)}\,dt \quad \text{and} \quad v_2 = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,dt$$

to find a particular solution for

$$t^2y'' - 4ty' + 4y = 3t^4 \ln t.$$

$$y'' - \frac{4}{t}y' + \frac{4}{t^2}y = 3t^2 \ln t$$

$y_1 = t$, $y_2 = t^4$ $\quad W[y_1, y_2] = \begin{vmatrix} t & t^4 \\ 1 & 4t^3 \end{vmatrix} = 4t^4 - t^4 = 3t^4$

$$V_1 = \int \frac{(-3t^2 \ln t)t^4}{3t^4}\,dt = \int (-t^2 \ln t)\,dt = -\left(\frac{t^3}{3}\ln t - \int \frac{t^2}{3}dt\right) = -\frac{t^3}{3}\ln t + \frac{t^3}{9}$$

$u = \ln t \quad dv = t^2\,dt$

$du = \frac{1}{t}\,dt \quad v = t^3/3$

$$V_2 = \int \frac{(3t^2 \ln t)t}{3t^4}\,dt = \int \frac{\ln t}{t}\,dt = \frac{(\ln t)^2}{2}$$

so $y_p = \frac{t^4}{9} - \frac{t^4}{3}\ln t + \frac{t^4}{2}(\ln t)^2$

(c) $\boxed{2}$ Find the general solution for

$$t^2y'' - 4ty' + 4y = 3t^4 \ln t.$$

$$y = C_1 t + C_2 t^4 + \frac{t^4}{9} - \frac{t^4}{3}\ln t + \frac{t^4}{2}(\ln t)^2$$

7. $\boxed{2}$ Let $f(t)$ be defined for $t \in [0, \infty)$, state the integral definition of the Laplace transform of $f(t)$.

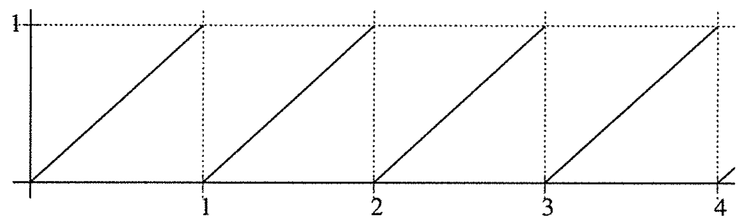
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

8. For each of the following functions, **use appropriate words to explain** if it has a Laplace transforms; do not compute the transform, if it exist.

(a) $\boxed{2}$ $\quad f(t) = \tan t$

f does not have a Laplace transform. It is neither piecewise continuous nor of exponential order.

(b) $\boxed{2}$ The sawtooth function defined by $f(t) = t$ for $t \in [0, 1)$ and $f(t+1) = f(t)$ for $t \geq 1$, i.e. $f(t)$ has period 1; see the figure below.



f does have a Laplace transform since it is piecewise continuous & of exponential order ($\alpha = 0$).

9. Find the Laplace transform of the following.

(a) $\boxed{4}$ $\quad f(t) = t^2 e^{3t}$

$$F(s) = \frac{2}{(s-3)^3}$$

(b) $\boxed{6}$ $\quad g(t) = t \sin t$ $\qquad$ [HINT: Use one of the provided properties.]

$$G(s) = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2}$$

10. Find the inverse Laplace transform of the following.

(a) $\boxed{4}$ $F(s) = \frac{23}{9-s} = \frac{-23}{s-9}$

$$f(t) = -23e^{9t}$$

(b) $\boxed{6}$ $G(s) = \frac{2s+2}{s^2-4s+13} = \frac{2(s-2)+2\cdot 3}{(s-2)^2+3^2}$

$$= 2e^{2t}\cos 3t + 2e^{2t}\sin 3t$$

(c) $\boxed{8}$ $Y(s) = \frac{3s^2-4s+21}{(s-3)(s^2+9)}$

$$\frac{3s^2-4s+21}{(s-3)(s^2+9)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+9}$$

$$3s^2-4s+21 = A(s^2+9) + (Bs+C)(s-3)$$

Let $s=3$

$27-12+21 = A(18)$ so $A=2$

Eq. Coeff

s^2 : $3 = A+B$ so $B=1$

s^0 : $21 = 9A-3C$ so $C=-1$

$$Y(s) = \frac{2}{s-3} + \frac{s}{s^2+9} - \frac{1}{s^2+9}$$

$$y(t) = 2e^{3t} + \cos 3t - \frac{1}{3}\sin 3t$$