1. Find the general solution for the following.

(a) \[ 2y'' + 3y' - 2y = 0 \]

\[
2r^2 + 3r - 2 = 0
\]
\[
(2r - 1)(r + 2) = 0
\]
\[
r = \frac{1}{2}, \quad r = -2
\]
\[
y = C_1 e^{\frac{t}{2}} + C_2 e^{-2t}
\]

(b) \[ y'' - 4y' + 13y = 0 \]

\[
r = \frac{4 \pm \sqrt{16 - 52}}{2}
\]
\[
r = \frac{4 \pm 6i}{2} = 2 \pm 3i
\]
\[
y = e^{2t} \cos 3t + e^{2t} \sin 3t
\]

2. \[ y'' - y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 7 \]

\[
r^2 - r - 6 = 0
\]
\[
(r - 3)(r + 2) = 0
\]
\[
y = C_1 e^{3t} + C_2 e^{-2t}
\]
\[
y(0) = C_1 + C_2 = 4
\]
\[
y'(0) = 3C_1 e^{3t} - 2C_2 e^{-2t}
\]
\[
y'(0) = 3C_1 - 2C_2 = 7
\]

So

\[
2C_1 + 2C_2 = 8
\]
\[
3C_1 - 2C_2 = 7
\]

\[
5C_1 = 15 \Rightarrow C_1 = 3, \quad C_2 = 1
\]
\[
y = 3e^{3t} + e^{-2t}
\]
3. Find the appropriate form using the Method of Undetermined Coefficients for a particular solution to the following. **Do not** solve for the unknown constants.

(a) \[ y'' - y = (t^2 - 1)e^t + 7 \]

\[ r^2 - 1 = 0 \]
\[ r = \pm 1 \]
\[ y_p = t \left( At^2 + Bt + C \right) e^t + 7 \]

(b) \[ y'' + 4y = 7t \cos 2t \]

\[ r^2 + 4 = 0 \]
\[ r = \pm 2i \]
\[ y_p = t \left( At + B \right) \cos 2t + t \left( Ct + D \right) \sin 2t \]

4. **10** Find the general solution to

\[ y'' - 2y' + y = 4(2t + 1)e^{3t}. \]

\[ (r - 1)^2 = 0 \]
\[ r = 1 \]
\[ y_1 = e^t \quad \& \quad y_2 = te^t \quad \text{solve } y'' - 2y' + y = 0. \]

Now, use MUC,

\[ y_p = (At + B)e^{3t} \]
\[ y_p' = \left[ A + 3At + 3B \right] e^{3t} \]
\[ y_p'' = \left[ 3A + 3At + 9A + 9B \right] e^{3t} \]

So

\[ y_p'' - 2y_p' + y_p = \left[ \left( 9A - 6A + A \right)t + \left( 6A + 9B - 2A - 6B + B \right) \right] e^{3t} \]
\[ = \left[ \left( 4A \right)t + \left( 4A + 4B \right) \right] e^{3t} = 4 \left[ At + (A+B) \right] e^{3t} \]

so \( A = 2, \ B = -1 \)

\[ y = C_1 e^t + C_2 te^t + (2t-1)e^{3t} \]
Let \( t > 0 \), for Cauchy-Euler equations of the form
\[
\alpha t^2 y'' + b t y' + c y = 0
\]  
we found that solutions were of the form \( y = t^r \) where \( r \) is a root of the characteristic equation
\[
\alpha r^2 + (b - \alpha) r + c = 0.
\]  
In the same section we discussed the Reduction of Order theorem. Specifically, let \( y_1 \) be a non-trivial solution, i.e. \( y_1 \neq 0 \), to
\[
y'' + p(t)y' + q(t)y = 0
\] on an interval \( I \). Then a second linearly independent solution to (3) is given by
\[
y_2(t) = y_1(t) \int \frac{e^{-g p(t)\, dt}}{y_1(t)^2} \, dt.
\]

5. Use Reduction of Order to show that if \( r \) is a double root of (2), and hence \( y_1 = t^r \) is a solution to (1), then a second linearly independent solution to (1) is given by \( y_2 = t^r \ln t \).

**HINT 1:** If \( r \) is a double root of (2) what can you say about \( (b - a)^2 - 4ac \)?
\[
(b-a)^2 - 4ac = 0
\]

**HINT 2:** With HINT 1 in mind, what can you say about \( r \) in terms of \( a \) and \( b \)?
\[
s_0 \quad r = \frac{- (b-2)}{2a} = \frac{a-b}{2a}
\]

**HINT 3:** Before you dive into the formula in (4) make sure you are using the form given in (3).

\[
y'' + \frac{b}{2a} y' + \frac{c}{2a} t^2 = 0 \quad \Rightarrow \quad e^{-\int p(t) \, dt} = e^{-\int \frac{b}{2a} \, dt} = e^{-\frac{b}{2a} \ln t} = t^{-\frac{b}{2a}}
\]

\[
y_2 = t^r \int \frac{t^{-\frac{b}{2a}} \, dt}{t} = t^r \int \frac{dt}{t^{1-\frac{b}{2a}}} = t^r \int \frac{dt}{t^{1-\frac{b}{2a}}} = t^r \ln t
\]
6. Consider the equation \( t^2y'' - 4ty' + 4y = 3t^4 \ln t \).

(a) \(4\) Find the general solution for
\[
t^2y'' - 4ty' + 4y = 0
\]
\[
\begin{align*}
  r^2 - 5r + 4 &= 0 \\
  (r - 4)(r - 1) &= 0
\end{align*}
\]
\[
\therefore \quad y = C_1 t + C_2 t^4.
\]

(b) \(14\) Use Variation of Parameters with
\[
v_1 = \int \frac{-g(t)y_2(t)}{W[y_1, y_2](t)} \, dt \quad \text{and} \quad v_2 = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)} \, dt
\]
to find a particular solution for
\[
t^2y'' - 4ty' + 4y = 3t^4 \ln t.
\]
\[
y'' - \frac{4}{t} y' + \frac{4}{t^2} y = 3t^2 \ln t
\]
\[
y_1 = t, \quad y_2 = t^4
\]
\[
W[y_1, y_2] = \begin{vmatrix} t & t^4 \\ 1 & 4t^3 \end{vmatrix} = 4t^4 - t^4 = 3t^4
\]
\[
V_1 = \int \frac{(-3t^5 \ln t \cdot t^4)}{3t^4} \, dt = \int (-t^6 \ln t) \, dt = -\left( \frac{t^3}{3} \ln t - \int \frac{t^2}{3} \, dt \right) = -\frac{t^3}{3} \ln t + \frac{t^3}{9}
\]
\[
u = \frac{t^3}{9}, \quad \frac{dv}{dt} = \frac{t^2}{3} \ln t, \quad \frac{du}{dt} = \frac{t^3}{3} \ln t
\]
\[
V_2 = \int \frac{(3t^2 \ln t)t}{3t^4} \, dt = \int \frac{\ln t}{t} \, dt = \frac{\ln t}{2}
\]
\[
\therefore \quad y_p = \frac{t^4}{9} - \frac{t^4}{3} \ln t + \frac{t^4}{2} \left( \ln t \right)^2
\]

(c) \(2\) Find the general solution for
\[
t^2y'' - 4ty' + 4y = 3t^4 \ln t.
\]
\[
y = C_1 t + C_2 t^4 + \frac{t^4}{9} - \frac{t^4}{3} \ln t + \frac{t^4}{2} \left( \ln t \right)^2
\]
7. Let $f(t)$ be defined for $t \in [0, \infty)$, state the integral definition of the Laplace transform of $f(t)$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

8. For each of the following functions, use appropriate words to explain if it has a Laplace transform; do not compute the transform, if it exist.

(a) $f(t) = \tan t$

$f$ does not have a Laplace transform. It is neither piecewise continuous, nor of exponential order.

(b) The sawtooth function defined by $f(t) = t$ for $t \in [0, 1)$ and $f(t+1) = f(t)$ for $t \geq 1$, i.e. $f(t)$ has period 1; see the figure below.

![Sawtooth Function](image)

$f$ does have a Laplace transform since it is piecewise continuous and of exponential order ($\alpha = 0$).

9. Find the Laplace transform of the following.

(a) $f(t) = t^2 e^{3t}$

$$\mathcal{L}\{f(t)\} = \frac{2}{(s-3)^3}$$

(b) $g(t) = t \sin t$ [HINT: Use one of the provided properties.]

$$\mathcal{L}\{g(t)\} = \frac{d}{ds} \left[ \frac{1}{s^2+1} \right] = \frac{2s}{(s^2+1)^2}$$
10. Find the inverse Laplace transform of the following.

(a) \( F(s) = \frac{23}{9-s} = \frac{-23}{s-9} \)

\[ f(t) = -23 e^{9t} \]

(b) \( G(s) = \frac{2s + 2}{s^2 - 4s + 13} = \frac{2(s-2) + 2.3}{(s-2)^2 + 3^2} \)

\[ = 2 e^{2t} \cos 3t + 2 e^{2t} \sin 3t \]

(c) \( Y(s) = \frac{3s^2 - 4s + 21}{(s-3)(s^2 + 9)} \)

\[ \frac{3s^2 - 4s + 21}{(s-3)(s^2 + 9)} = \frac{A}{s-3} + \frac{Bs + C}{s^2 + 9} \]

\[ 3s^2 - 4s + 21 = A(s^2 + 9) + (Bs + C)(s-3) \]

Let \( s = 3 \)

\[ 24 - 12 + 21 = A \left( 18 \right) \quad \text{so} \quad A = 2 \]

For \( C = \frac{7}{9} \)

\[ s_i : \quad 3 = A + B \quad \Rightarrow \quad B = 1 \]

\[ s_o : \quad 21 = 9A - 3C \quad \Rightarrow \quad C = -1 \]