Math 274 Exam 1 **Name:** ______________________

21 May 2018 Point values in $\boxed{\text{boxes}}$.

1. For $b > 0$, consider the equation
$$y' = by.$$

(a) $\boxed{5}$ The equation is separable, find a general solution using that method.

$$\frac{dy}{dt} = by$$

$$\int \frac{dy}{y} = \int b\,dt$$

$$\ln|y| = bt + C \quad \text{or} \quad y \equiv 0$$

$$\text{or} \quad y = Ce^{bt}$$

(b) $\boxed{5}$ The equation is first order linear, find a general solution using an integrating factor.

$$y' - by = 0 \qquad \mu(t) = e^{\int -b\,dt} = e^{-bt}$$

$$(ye^{-bt})' = 0$$

$$ye^{-bt} = C$$

$$y = Ce^{bt}$$

(c) $\boxed{5}$ The equation is constant coefficient linear, use the characteristic equation (auxiliary equation) to find a general solution.

$$y' - by = 0$$

$$r - b = 0$$

$$r = b$$

$$y = Ce^{bt}$$

2. For each of the following first order equations determine if it is Separable, Linear, Homogeneous, and/or Bernoulli. Check each box that applies. (More than one may apply.)

(a) $\boxed{4}$ $\frac{ds}{dt} = 5 - \frac{s}{10}$

Separable	Linear	Homogeneous	Bernoulli
✓	✓		✓

All linear equations are Bernoulli

$\frac{ds}{dt} = \frac{50-s}{10}$

$\frac{ds}{dt} + \frac{s}{10} = 5$

(b) $\boxed{4}$ $\frac{ds}{dt} = 5 - \frac{s}{10-t}$

Separable	Linear	Homogeneous	Bernoulli
	✓		✓

$\frac{ds}{dt} + \frac{s}{10-t} = 5$

(c) $\boxed{4}$ $x^2\frac{dx}{dt} = t^2 - \frac{x^3}{t}$

Separable	Linear	Homogeneous	Bernoulli
		✓	✓

$\frac{dx}{dt} = \frac{t^2}{x^2} - \frac{x}{t}$

$\frac{dx}{dt} + \frac{x}{t} = t^2 x^{-2}$

(d) $\boxed{4}$ $\frac{dy}{dt} = t^2y^4 + 6t^2y$

Separable	Linear	Homogeneous	Bernoulli
✓			✓

$\frac{dy}{dt} = t^2\left(y^4 + 6y\right)$

$\frac{dy}{dt} - 6t^2 y = t^2 y^4$

3. $\boxed{10}$ Use an appropriate substitution to find a general solution to

$$\frac{dy}{dx} = \frac{y + 2xe^{-y/x}}{x}.$$

$$\frac{dy}{dx} = \frac{y}{x} + 2e^{-y/x}$$

Let $v = \frac{y}{x}$ so $xv = y$ & $v + x\frac{dv}{dx} = \frac{dy}{dx}$.

$$v + x\frac{dv}{dx} = v + 2e^{-v}$$

$$\int e^{v}\,dv = \int \frac{2}{x}\,dx$$

$$e^{v} = 2\ln|x| + C$$

so $e^{y/x} = 2\ln|x| + C$

is an implicit solution,

$$y = x\ln\left(2\ln|x| + C\right)$$

is an explicit solution.

4. $\boxed{6}$ Verify $y = 2te^{2t}$ solves the nonhomogeneous equation

$$y'' - y' - 2y = 6e^{2t}.$$

$$y = 2te^{2t}$$

$$y' = 2e^{2t} + 4te^{2t}$$

$$y'' = 4e^{2t} + 4e^{2t} + 8te^{2t}$$

Substituting gives

$$y'' - y' - 2y = 8e^{2t} + 8te^{2t} - 2e^{2t} - 4te^{2t} - 4te^{2t} = 6e^{2t}$$

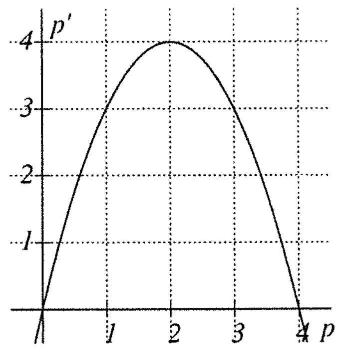
5. $\boxed{4}$ Find the interval that a unique solution to the following initial value problem exists.

$$(t^2 - 16)y' - y = \frac{\sin t}{t+1}, \qquad y(0) = 57$$

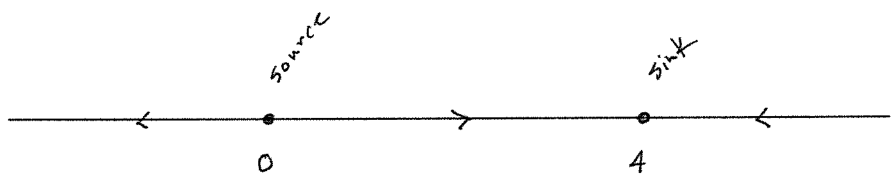
$$y' - \frac{y}{t^2 - 16} = \frac{\sin t}{(t^2-16)(t+1)}$$ ← Discontinuities at $t = -4, -1, 4$

$$t \in (-1, 4)$$

6. Consider a population model of bison on the Great Plains given by $\dfrac{dp}{dt} = p(4-p)$ where p is measured in millions, see figure.



(a) $\boxed{4}$ Sketch the phase line for the model. Label each equilibrium as a sink, source, or node. (Sketch your phase line horizontally for convenience with $-\infty$ on the left and $+\infty$ on the right, as usual.)

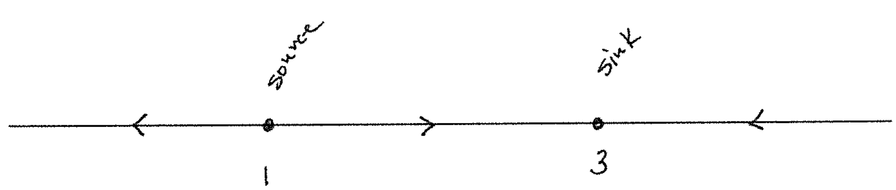


(b) $\boxed{3}$ What is the carrying capacity of the system, i.e. what is the asymptotic behavior of a population satisfying $p(0) > 0$ as $t \to \infty$.

$$p \longrightarrow 4 \quad \text{as} \quad t \longrightarrow \infty$$

7. If a constant harvesting rate is introduced into the bison population model from above we have $\dfrac{dp}{dt} = p(4-p) - 3$, where p is measured in millions.

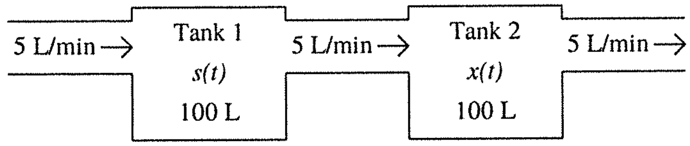
(a) $\boxed{4}$ Sketch the phase line for the model with harvesting. Label each equilibrium as a sink, source, or node.



(b) $\boxed{3}$ What initial populations, i.e. $p(0) = p_0 > 0$, will lead to extinction with constant harvesting?

$$p(0) \in (0, 1)$$

8. Consider a two tank system as in the figure.



(a) $\boxed{6}$ The first tank is initially filled with 100 L of pure water. A brine solution containing 0.2 kg of salt per liter is being pumped into the tank at a rate of 5 L/min. The tank is well mixed and drains at 5 L/min. Let $s(t)$ be the amount of salt (in kg) in the tank at time t (in min).
Set up an initial value problem (a differential equation with initial data) modeling $s(t)$. **DO NOT SOLVE.**

$$\frac{ds}{dt} = 5(0.2) - 5\left(\frac{s}{100}\right), \qquad s(0) = 0$$

(b) $\boxed{10}$ A second tank is initially filled with 100 L of a brine solution with a concentration of 0.3 kg/L. The solution flowing out of the first tank flows into the second tank at 5 L/min. The second tank is also well stirred and drains at 5 L/min. The amount of salt $x(t)$ (in kg) in the tank at time t (in min) is modeled by the initial value problem

$$\frac{dx}{dt} = 1 - e^{-t/20} - \frac{x}{20}, \qquad x(0) = 30.$$

Solve the initial value problem.

$$\frac{dx}{dt} + \frac{x}{20} = 1 - e^{-t/20} \qquad \mu(t) = e^{t/20}$$

$$\left(x e^{t/20}\right)' = e^{t/20} - 1$$

$$x e^{t/20} = \int \left(e^{t/20} - 1\right) dt = 20 e^{t/20} - t + C$$

$$x = 20 + e^{-t/20}(C - t)$$

$$x(0) = 30 \implies C = 10, \text{ so}$$

$$x(t) = 20 + e^{-t/20}(10 - t)$$

9. $\boxed{5}$ Evaluate.

$$\text{Im}\left((3+2i)e^{(i\pi/3)}\right)$$

$$= \text{Im}\left[(3+2i)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$$

$$= 3\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3} = \frac{3\sqrt{3}}{2} + 1$$

10. $\boxed{8}$ Find a general solution to the following equations.

(a) $y'' - 9y = 0$

$r^2 - 9 = (r-3)(r+3) = 0$

$y = C_1 e^{3t} + C_2 e^{-3t}$

(b) $x'' = 6x' - 9x$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$ $\qquad y = C_1 e^{3t} + C_2 t e^{3t}$

11. $\boxed{6}$ Find the real-valued solution to the initial value problem

$$y'' + 2y' + 10y = 0, \qquad y(0) = 2, y'(0) = 1.$$

$r^2 + 2r + 10 = 0$

$(r+1)^2 + 3^2 = 0$

$r = -1 \pm 3i$

$y = C_1 e^{-t}\cos 3t + C_2 e^{-t}\sin 3t$

$y(0) = 2 \Rightarrow C_1 = 2$

$y' = -2e^{-t}\cos 3t + 6e^{-t}\sin 3t - C_2 e^{-t}\sin 3t + 3C_2 e^{-t}\cos 3t$

$y'(0) = 1 \Rightarrow C_2 = 1$, so

$y = 2e^{-t}\cos 3t + e^{-t}\sin 3t$ is the solution.

Page	1	2	3	4	5	6	Total
Value	15	16	20	14	16	19	100
Points							