Math 274 Exam 2 **Name:** ____________________

29 May 2018

Point values in [boxes].

1. [5] Solve the initial value problem

$$y' = by, \qquad y(0) = 1$$

using the method of Laplace transforms.

$y' - by = 0$

$sY - 1 - bY = 0$

$Y(s-b) = 1$

$Y = \frac{1}{s-b}$

so $y(t) = e^{bt}$

2. [3] Find a general solution to

$$y'' - y' - 6y = 0.$$

$r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$r_1 = 3, \; r_2 = -2$

$y = C_1 e^{3t} + C_2 e^{-2t}$

3. Find the appropriate form using the Method of Undetermined Coefficients for a particular solution to the following. **Do not** solve for the unknown constants.

(a) [3] $\quad y'' - y' - 6y = (3t+2)e^{3t}$

$y_p = t(At+B)e^{3t}$

(b) [3] $\quad y'' - y' - 6y = 8 - 4e^{t}$

$y_p = A + Be^{t}$

(c) [3] $\quad y'' - y' - 6y = 3t \sin t$

$y_p = (At+B)\cos t + (Ct+D)\sin t$

(d) [3] $\quad y'' - y' - 6y = e^{-2t}\cos 3t$

$y_p = Ae^{-2t}\cos 3t + Be^{-2t}\sin 3t$

4. For $-1 < x < 1$, consider the Legendre equation

$$(1-x^2)y'' - 2xy' + 2y = 0. \tag{1}$$

(a) [3] Verify $y_1 = x$ is a solution to (1).

$y_1' = 1$

$y_1'' = 0$

(b) [12] Find a general solution to (1).

standard form: $y'' + \underbrace{\frac{-2x}{1-x^2}}_{p(x)} y' + \frac{2}{1-x^2} y = 0$

$e^{-\int p(x)\,dx} = e^{\int \frac{2x}{1-x^2}\,dx} = e^{-\ln|1-x^2|} = \frac{1}{1-x^2}$

$y_p = x \int \frac{dx}{x^2(1-x^2)}$

$\frac{1}{x^2(1-x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}$

$1 = Ax(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$

$x=0: \quad B = 1 \qquad x = 1: \quad C = \frac{1}{2} \qquad x = -1: \quad D = \frac{1}{2} \qquad x': \quad 0 = A$

so

$y_p = x \int \left(\frac{1}{x^2} + \frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x} \right) dx$

$= x \left(-\frac{1}{x} - \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| \right)$

$= -1 + \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right).$

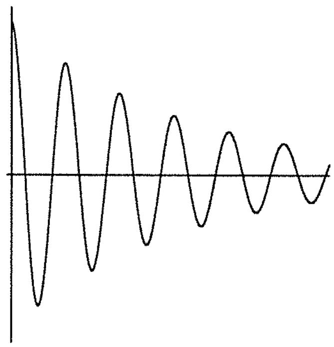
Finally, a general solution is

$y_p = C_1 x + C_2 \left(\frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1 \right)$

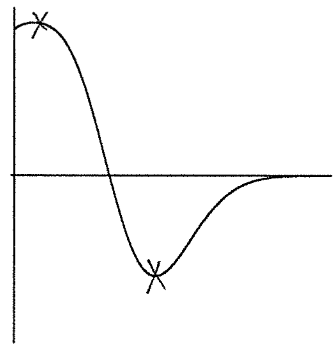
5. For $m, k > 0, b \geq 0$, the standard Mass-Spring system is given by $mx'' + bx' + kx = 0$.

(a) [1] Qualitatively, the behavior of a solution to a **Critically Damped system** is similar to an **Underdamped** / **Overdamped** system. (Circle One.)

(b) [9] The figures below show possible solution curves to this system. Label the system as **Undamped**, **Under**damped, **Over**damped, and/or **Not** a system of this form.



- Undamped
- Over
- Under
- Not



- Undamped
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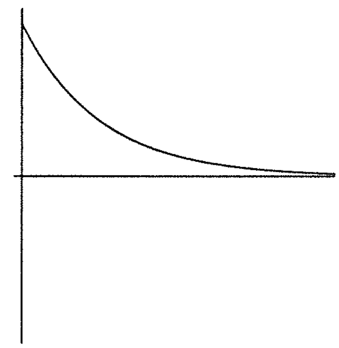

- Undamped
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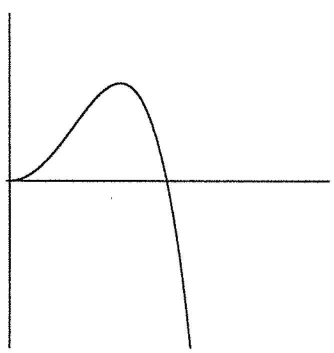

- Undamped
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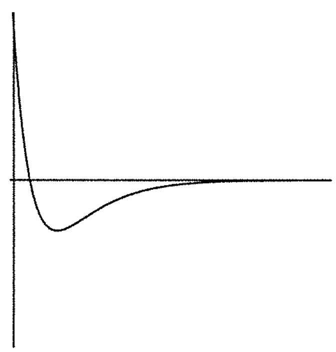

- Undamped
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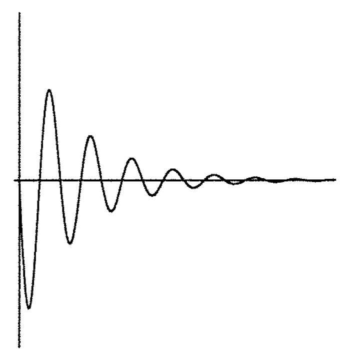

- Undamped
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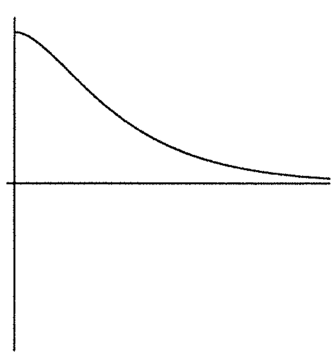

- Undamped
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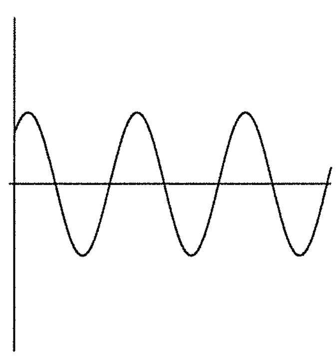

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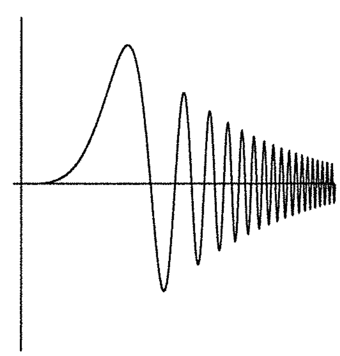

- Undamped
- Over
- Under
- Not

6. $\boxed{10}$ Solve the initial value problem

$$y'' - 3y' + 2y = 2e^{3t}, \qquad y(0) = 6, y''(0) = 10.$$

HINT: We discussed three methods for solving initial value problems of this form, one of those methods is much easier than the other two.

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$y = C_1 e^t + C_2 e^{2t}$

$y_p = Ae^{3t}$

$y_p' = 3Ae^{3t}$

$y_p'' = 9Ae^{3t}$

$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = 2e^{3t}$

so $A = 1$

$y = C_1 e^t + C_2 e^{2t} + e^{3t}$

$y(0) = C_1 + C_2 + 1 = 6$

$y' = C_1 e^t + 2C_2 e^{2t} + 3e^{3t}$

$y'(0) = C_1 + 2C_2 + 3 = 10$

$C_1 + C_2 + 1 = 6$

$C_2 + 2 = 4$ so $C_2 = 2$, $C_1 = 3$

$y = 3e^t + 2e^{2t} + e^{3t}$

7. $\boxed{10}$ Use the definition to determine the Laplace transform of $f(t) = \begin{cases} 0, & 0 \le t < 4 \\ e^{3t}, & 4 < t \end{cases}$.

Specify the domain on which the transform is defined.

$$F(s) = \int_4^\infty e^{-st} e^{3t}\, dt = \int_4^\infty e^{-(s-3)t}\, dt$$

$$= \lim_{R\to\infty} \left[\frac{-1}{s-3} e^{-(s-3)t} \right] \Bigg|_4^R$$

$$= \frac{e^{-4(s-3)}}{s-3} - \underbrace{\lim_{R\to\infty} \frac{e^{-(s-3)R}}{s-3}}_{=0 \text{ iff } s-3>0\text{, i.e. } s>3}$$

so $F(s) = \dfrac{e^{12-4s}}{s-3}$ for $s > 3$.

8. $\boxed{10}$ Apply the Laplace transform to the initial value problem

$$4y'' - y = e^{2t}, \qquad y(0) = 2, y'(0) = -1$$

to express $Y(s) = \mathscr{L}\{y(t)\}$ in the form $Y(s) = \frac{P(s)}{Q(s)}$; for example, (2) below is of this form. In particular, express $P(s)$ as a single polynomial, i.e. multiply out and combine like terms. Express $Q(s)$ factored into linear and/or irreducible quadratic terms.

Do not find the inverse Laplace transform.

$$4\left(s^2 Y - 2s + 1\right) - Y = \frac{1}{s-2}$$

$$Y\left(4s^2 - 1\right) = \frac{1}{s-2} + 8s - 4 = \frac{1}{s-2} + \frac{(8s-4)(s-2)}{s-2}$$

$$Y = \frac{8s^2 - 20s + 9}{(s-2)(2s-1)(2s+1)}$$

9. $\boxed{12}$ Applying the Laplace transform to the initial value problem

$$y'' + 2y' + 5y = 12e^{-t}, \qquad y(0) = 5, y'(0) = -6$$

gives the following

$$Y(s) = \frac{5s^2 + 9s + 16}{(s+1)(s^2 + 2s + 5)}. \tag{2}$$

Determine $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, the solution to the given initial value problem.

$$\frac{5s^2 + 9s + 16}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s^2 + 9s + 16 = A(s^2 + 2s + 5) + (Bs + C)(s+1)$$

$s = -1$: $5 - 9 + 16 = A(4)$ so $A = 3$

s^2 : $5 = A + B$ so $B = 2$

s^0 : $16 = 5A + C$ so $C = 1$

$$y = 3e^{-t} + 2e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$Y = \frac{3}{s+1} + \frac{2(s+1)}{(s+1)^2 + 2^2} - \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

10. For $x > 0$, consider the differential equation

$$xy'' - (1+x)y' + y = x^2 e^{2x}.$$

(a) [5] Both $y_1 = e^x$ and $y_2 = (1+x)$ are solutions to the associated homogeneous equation, show that they are linearly independent.

$$W[e^x, 1+x] = \begin{vmatrix} e^x & 1+x \\ e^x & 1 \end{vmatrix} = e^x - (e^x + xe^x) = -xe^x \neq 0$$

for $x > 0$

(b) [8] Set up the variation of parameters expression for the particular solution to the original inhomogeneous equation. Reasonably simplify the integrals but **DO NOT EVALUATE**.

Std Form: $y'' - \frac{1+x}{x} y' + \frac{1}{x} y = xe^{2x}$

$$y_p = e^x \int \frac{-xe^{2x}(1+x)}{-xe^x} dx + (1+x) \int \frac{xe^{2x}(e^x)}{-xe^x} dx$$

$$= e^x \int e^x (1+x)\, dx - (1+x) \int e^{2x}\, dx$$

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Value	10 ~~25~~	15	10	20	22	13	100
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