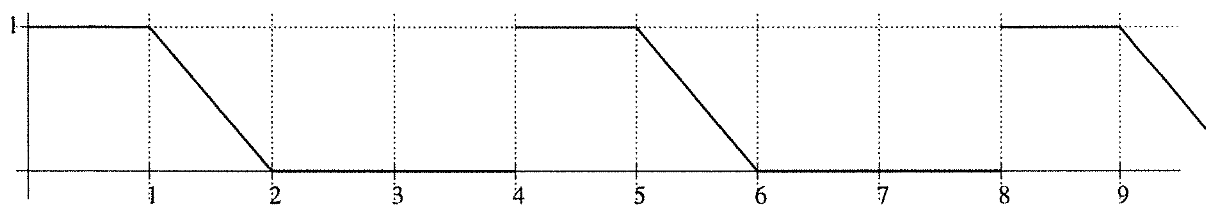
Math 274 Exam 3 **Name:** ______________________

4 June 2018 Point values in $\boxed{\text{boxes}}$.

1. $\boxed{10}$ A graph of $y = f(t)$ is given below.



Find the Laplace transform of $f(t)$.

$$f_4 = 1 + u(t-1)\left[2 - t - 1\right] + u(t-2)(t-2)$$

$$F_4 = \frac{1}{s} + e^{-s}\left(-\frac{1}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2}\right)$$

$$F = \frac{s - e^{-s} + e^{-2s}}{s\left(1 - e^{-4s}\right)}$$

2. $\boxed{15}$ Solve the initial value problem.

$$y'' + y = f(t), \qquad y(0) = 1, y'(0) = 0$$

where

$$f(t) = \begin{cases} 2e^t, & t < 1 \\ 0, & 1 < t \end{cases} = 2e^t - u(t-1)\,2e^t$$

You will find it useful to know that

$$\frac{2}{(s-1)(s^2+1)} = \frac{1}{s-1} - \frac{s+1}{s^2+1}.$$

$$s^2 Y - s + Y = \frac{2}{s-1} - 2e^{-s}\mathcal{L}\{e^{t+1}\}$$

$$Y(s^2+1) = \frac{2}{s-1} - e^{1-s} \cdot \frac{2}{s-1} + s$$

$$Y = \frac{2}{(s-1)(s^2+1)} - e^{1-s} \cdot \frac{2}{(s-1)(s^2+1)} + \frac{s}{s^2+1}$$

$$y = e^{+t} - \cos t - \sin t - e\,u(t-1)\left[e^{+(t-1)} - \cos(t-1) - \sin(t-1)\right] + \cos t$$

$$= e^{+t} - \sin t - e\,u(t-1)\left[e^{t-1} - \cos(t-1) - \sin(t-1)\right]$$

$$= e^t - \sin t - u(t-1)\left[e^t - e\left(\cos(t-1) + \sin(t-1)\right)\right]$$

3. Assume $g(t)$ is piecewise continuous and of exponential order and consider the initial value problem

$$y'' + 4y' + 5y = g(t), \qquad y(0) = 0, y'(0) = 0.$$

(a) $\boxed{10}$ Find the solution. Express your solution in terms of a convolution.

$$s^2 Y + 4sY + 5Y = G(s)$$

$$Y = G(s) \cdot \frac{1}{(s+2)^2 + 1}$$

$$y = g(t) * e^{-2t} \sin t$$

(b) $\boxed{5}$ If $g(t) = e^{-2t}$, find the solution, i.e., evaluate the convolution.

$$y = \int_0^t e^{-2t+2v} \cdot e^{-2v} \sin v \, dv = e^{-2t} \int_0^t \sin v \, dv$$

$$= e^{-2t} (-\cos v) \Big|_0^t = e^{-2t} (1 - \cos t)$$

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4. $\cancel{\boxed{10}}$ Find the solution to the symbolic second order initial value problem

$$y'' + 4y' + 3y = 2\delta(t-1), \qquad y(0) = 40, y'(0) - 60.$$

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Express your solutions as a piecewise defined function.

You will find it useful to know that

$$\frac{40s+100}{s^2+4s+3} = \frac{30}{s+1} + \frac{10}{s+3}.$$

$$s^2Y - 40s + 60 + 4sY - 160 + 3Y = 2e^{-s}$$

$$Y(s^2 + 4s + 3) = 2e^{-s} + 40s + 100$$

$$Y = e^{-s} \cdot \frac{2}{(s+1)(s+3)} + \frac{40s+100}{(s+1)(s+3)}$$

$$\frac{2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$2 = A(s+3) + B(s+1)$$

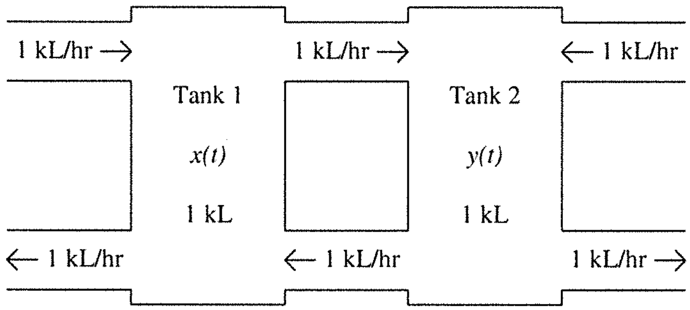
$s = -1: \ A = 1 \qquad s = -3: \ B = -1$

$$Y = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+3}\right) + \frac{30}{s+1} + \frac{10}{s+3}$$

$$y = u(t-1)\left[e^{-(t-1)} - e^{-3(t-1)}\right] + 30e^{-t} + 10e^{-3t}$$

$$y = \begin{cases} 30e^{-t} + 10e^{-3t} & t < 1 \\ (30+e)e^{-t} + (10-e^3)e^{-3t} & t > 1 \end{cases}$$

5. $\boxed{5}$ Tank 1 initially contains 20 kg of salt dissolved into 1 kL of water. Tank 2 initially contains 40 kg of salt dissolved into 1 kL of water. Both tanks are well mixed. Pure water is flowing into each tank at the rate specified in the figure. Similarly, the figure shows the rate the mixtures are flowing between each tank and being drained. Let $x(t)$ be the amount of salt in tank 1 in kg, and $y(t)$ be the amount of salt in tank 2 in kg.



After an hour Jode Ummass accidentally knocks a bag of salt into tank 1. The amount of salt in each tank is modeled by the symbolic system

$$x' = -2x + y + 2\delta(t-1), \qquad x(0) = 20$$
$$y' = x - 2y, \qquad y(0) = 40.$$

Using the substitution $x = y' + 2y$ converts the system to the symbolic second order initial value problem

$$y'' + 4y' + 3y = 2\delta(t-1), \qquad y(0) = 40, y'(0) = -60.$$

Find the amount of salt in tank 1 for $t \in (0,1)$ and for $t \in (1,\infty)$. Express your solution as a piecewise defined function. **See the previous question!**

$$y = \begin{cases} 30e^{-t} + 10e^{-3t} & t < 1 \\ (30+e)e^{-t} + (10-e^3)e^{-3t} & t > 1 \end{cases}$$

$$y' = \begin{cases} -30e^{-t} - 30e^{-3t} & t < 1 \\ -(30+e)e^{-t} - 3(10-e^3)e^{-3t} & t > 1 \end{cases}$$

$$x = y' + 2y = \begin{cases} 30e^{-t} - 10e^{-3t} & t < 1 \\ (30+e)e^{-t} - (10-e^3)e^{-3t} & t > 1 \end{cases}$$

6. Consider the mass-spring system given by the initial value problem

$$x'' + 9x = 0, \qquad x(0) = 1, x'(0) = 3. \tag{1}$$

(a) $\boxed{5}$ Find the solution to (1).

$$s^2 X - s - 3 + 9X = 0$$

$$X(s^2+9) = s+3$$

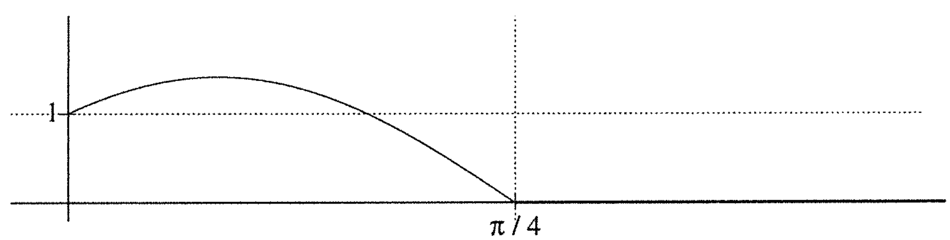
$$X = \frac{s+3}{s^2+9}$$

$$x = \cos 3t + \sin 3t$$

(b) $\boxed{5}$ Find the impulse needed to stop the motion of the system when it first returns to equilibrium at time $t_1 = \pi/4$, i.e., find M so that the solution to the symbolic initial value problem

$$x'' + 9x = M\delta(t - \pi/4), \qquad x(0) = 1, x'(0) = 3$$

has the following graph.



$$x' = -3\sin 3t + 3\cos 3t$$

$$x'\left(\frac{\pi}{4}\right) = -3\left(\frac{1}{\sqrt{2}}\right) + 3\left(-\frac{1}{\sqrt{2}}\right) = \frac{-6}{\sqrt{2}} = -3\sqrt{2}$$

so $M = 3\sqrt{2}$

7. $\boxed{5}$ Consider the system given by

$$x' = 2x + 2y + 6, \qquad x(0) = 1$$
$$y' = x + 4y + e^{2t}, \qquad y(0) = 1.$$

Convert the system into a second order initial value problem in **standard form** in y. **Do not solve.**

$$x = y' - 4y - e^{2t}$$

$$x' = y'' - 4y' - 2e^{2t}$$

$$y'' - 4y' - 2e^{2t} = 2y' - 8y - 2e^{2t} + 2y + 6$$

$$y'' - 6y' + 6y = 6 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 6$$

8. $\boxed{5}$ Write the following third order equation as a system of first order equations in matrix notation.

$$y''' - y'' + 3y' - 5y = 0$$

$x_1 = y \qquad x_1' = y' = x_2$

$x_2 = y' \qquad x_2' = y'' = x_3$

$x_3 = y'' \qquad x_3' = y''' = 5y - 3y' + y''$

$\qquad = 5x_1 - 3x_2 + x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

9. $\boxed{5}$ Verify that the given vector function satisfies the given system.

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$$

$$\vec{x}\,' = \begin{bmatrix} 3e^{3t} \\ 6e^{3t} \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 6e^{3t} \end{bmatrix}$$

10. Consider the following matrices

$$\mathbf{A}=\begin{bmatrix}1&0&1\\1&3&2\\0&2&1\end{bmatrix},\mathbf{B}=\begin{bmatrix}-4&4&-4\\-2&2&-2\\5&-4&5\end{bmatrix},\text{ and }\mathbf{C}=\begin{bmatrix}-1&2&-3\\-1&1&-1\\2&-2&3\end{bmatrix}.$$

(a) $\boxed{5}$ Compute the product **AB**.

$$\mathbf{AB}=\begin{bmatrix}1&0&1\\1&3&2\\0&2&1\end{bmatrix}\begin{bmatrix}-4&4&-4\\-2&2&-2\\5&-4&5\end{bmatrix}=\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}$$

(b) $\boxed{5}$ Compute the product **AC**.

$$\mathbf{AC}=\begin{bmatrix}1&0&1\\1&3&2\\0&2&1\end{bmatrix}\begin{bmatrix}-1&2&-3\\-1&1&-1\\2&-2&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(c) $\boxed{5}$ Solve the system of linear equations.

$$\begin{aligned}a\qquad+c&=2\\a+3b+2c&=3\\2b+c&=-1\end{aligned}$$

$$A\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2\\3\\-1\end{bmatrix}\quad\text{so}\quad\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-1&2&-3\\-1&1&-1\\2&-2&3\end{bmatrix}\begin{bmatrix}2\\3\\-1\end{bmatrix}\qquad\text{since } C=A^{-1}$$

$$=\begin{bmatrix}7\\2\\-5\end{bmatrix}$$

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