Math 274 Homework **Name:** ____________

Sections: 9.4 Point Values in $\boxed{\text{boxes}}$.

Due: 5 June 2018

1. $\boxed{5}$ Verify

$$\left\{ \begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix}, \begin{bmatrix} -2e^{2t} \\ e^{2t} \\ 4e^{2t} \end{bmatrix}, \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 4e^{3t} \end{bmatrix} \right\}$$

$\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$

is a fundamental solution set to

$$\mathbf{x}'(t) = \underset{A}{\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}} \mathbf{x}(t).$$

Recall a fundamental solution set for a system of n first order equations is a set of n linearly independent solutions. You should show each vector function is a solution. You should also show that they are linearly independent.

$$\vec{x}_1{}' = \begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix} \quad \checkmark \quad A\vec{x}_1 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix} = \begin{bmatrix} -e^t \\ e^t \\ 2e^t \end{bmatrix}$$

$$\vec{x}_2{}' = \begin{bmatrix} -4e^{2t} \\ 2e^{2t} \\ 8e^{2t} \end{bmatrix} \quad \checkmark \quad A\vec{x}_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -2e^{2t} \\ e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ 2e^{2t} \\ 8e^{2t} \end{bmatrix}$$

$$\vec{x}_3{}' = \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \\ 12e^{3t} \end{bmatrix} \quad \checkmark \quad A\vec{x}_3 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -e^{3t} \\ e^{3t} \\ 4e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \\ 12e^{3t} \end{bmatrix}$$

$$W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = \begin{vmatrix} -e^t & -2e^{2t} & -e^{3t} \\ e^t & e^{2t} & e^{3t} \\ 2e^t & 4e^{2t} & 4e^{3t} \end{vmatrix} = -e^t\left(4e^{5t} - 4e^{5t}\right) + 2e^{2t}\left(4e^{4t} - 2e^{4t}\right) - e^{3t}\left(4e^{3t} - 2e^{3t}\right)$$

$$= 4e^{6t} - 2e^{6t} = 2e^{6t} \neq 0$$

so the set is a fundamental solution set.

2. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$.

(a) $\boxed{3}$ Show

$$\left\{ \overbrace{\begin{bmatrix} 2e^{-2t}\cos t \\ -e^{-2t}(\cos t + \sin t) \end{bmatrix}}^{\vec{x}_1}, \overbrace{\begin{bmatrix} 2e^{-2t}\sin t \\ e^{-2t}(\cos t - \sin t) \end{bmatrix}}^{\vec{x}_2}, \right\}$$

is a fundamental solution set for $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\vec{x}_1{}' = \begin{bmatrix} -4e^{-2t}\cos t - 2e^{-2t}\sin t \\ 2e^{-2t}(\cos t + \sin t) - e^{-2t}(\cos t - \sin t) \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-2t}(2\cos t + \sin t) \\ e^{-2t}(\cos t + 3\sin t) \end{bmatrix} \checkmark$$

$$A\vec{x}_1 = \begin{bmatrix} -2e^{-2t}\cos t - 2e^{-2t}(\cos t + \sin t) \\ -2e^{-2t}\cos t + 3e^{-2t}(\cos t + \sin t) \end{bmatrix}$$

$$\vec{x}_2{}' = \begin{bmatrix} -4e^{-2t}\sin t + 2e^{-2t}\cos t \\ -2e^{-2t}(\cos t - \sin t) + e^{-2t}(-\sin t - \cos t) \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t}(\cos t - 2\sin t) \\ e^{-2t}(\sin t - 3\cos t) \end{bmatrix} \checkmark$$

$$A\vec{v}_2 = \begin{bmatrix} -2e^{-2t}\sin t + 2e^{-2t}(\cos t - \sin t) \\ -2e^{-2t}\sin t - 3e^{-2t}(\cos t - \sin t) \end{bmatrix}$$

$$W[\vec{x}_1, \vec{x}_2] = \begin{vmatrix} 2e^{-2t}\cos t & 2e^{-2t}\sin t \\ -e^{-2t}(\cos t + \sin t) & e^{-2t}(\cos t - \sin t) \end{vmatrix} = 2e^{-4t}(\cos^2 t - \cos t \sin t)$$

$$+ 2e^{-4t}(\cos t \sin t + \sin^2 t)$$

$$= 2e^{-4t} \neq 0 \text{ for all } t.$$

(b) $\boxed{1}$ Find a fundamental matrix for $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

$$\underline{X} = \begin{bmatrix} 2e^{-2t}\cos t & 2e^{-2t}\sin t \\ -e^{-2t}(\cos t + \sin t) & e^{-2t}(\cos t - \sin t) \end{bmatrix}$$

$$\underline{X}(0) = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \qquad \underline{X}^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

(c) $\boxed{1}$ Solve the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

$$\vec{x} = \underline{X} \cdot \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \underline{X}\begin{bmatrix} 1/2 \\ 9/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t}(\cos t + 9\sin t) \\ e^{-2t}(4\cos t - 5\sin t) \end{bmatrix}$$