**Math 274 In-Class** **Worksheet 10p**

Section: 7.8

1. Assume $g(t)$ is piecewise continuous and of exponential order and consider the initial value problem

$$y'' + 9y = g(t), \qquad y(0) = 1, y'(0) = 6.$$

Find the solution. Express your solution in terms of a convolution.

$$s^2 Y - s - 6 + 9Y = G$$

$$Y = G\,\frac{1}{(s^2+9)} + \frac{s}{s^2+9} + \frac{6}{s^2+9}$$

$$Y(s^2+9) = G + s + 6$$

$$y = g(t) * \frac{1}{3}\sin 3t + \cos 3t + 2\sin 3t$$

2. Assume $g(t)$ is piecewise continuous and of exponential order and consider the initial value problem

$$y'' - 3y' + 2y = g(t), \qquad y(0) = 1, y'(0) = 1.$$

(a) Find the solution. Express your solution in terms of a convolution.

$$s^2 Y - s - 1 - 3sY + 3 + 2Y = G$$

$$Y(s^2 - 3s + 2) = G + s - 2$$

$$y = g * (e^{2t} - e^t) + e^t$$

$$Y = G\left[\frac{1}{s^2-3s+2}\right] + \frac{1}{s-1}$$

$$= G\left[\frac{1}{s-2} - \frac{1}{s-1}\right] + \frac{1}{s-1}$$

(b) Express the convolution in (a) as an appropriate integral.

$$(g * (e^{2t} - e^t)) = \int_0^t g(t-v)(e^{2v} - e^v)\,dv$$

(c) If $g(t) = e^t$, evaluate the convolution.

$$e^t * (e^{2t} - e^t) = \int_0^t e^{t-v}(e^{2v} - e^v)\,dv = e^t \int_0^t (e^v - 1)\,dv$$

$$= e^t (e^v - v)\Big|_0^t = e^t(e^t - t - 1)$$

3. Let $F(s) = \dfrac{5}{s^2 - s - 6}$.

(a) Use partial fractions to find the inverse Laplace transform $f(t)$.

$$\frac{5}{s^2-s-6} = \frac{1}{s-3} - \frac{1}{s+2}$$

$$f(t) = e^{3t} - e^{-2t}$$

(b) Use the convolution theorem to find the inverse Laplace transform $f(t)$.

$$F(s) = \left(\frac{5}{s-3}\right)\left(\frac{1}{s+2}\right)$$

$$f(t) = 5e^{3t} * e^{-2t}$$

(c) Compute the convolution integral to show your solutions are equivalent.

$$f(t) = \int_0^t 5e^{3t-3v} e^{-2v}\,dv = 5e^{3t}\int_0^t e^{-5v}\,dv = 5e^{3t}\left(-\frac{1}{5}e^{-5v}\right)\Big|_0^t$$

$$= 5e^{3t}\left(-\frac{1}{5}e^{-5t} + \frac{1}{5}\right) = e^{3t} - e^{-2t} \checkmark$$

4. Compute the inverse Laplace transform of $F(s) = \dfrac{1}{(s^2+4)^2}$.

Note: $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ is a useful identity.

$$F(s) = \frac{1}{4}\cdot\frac{2}{s^2+4}\cdot\frac{2}{s^2+4}$$

$$f(t) = \frac{1}{4}\sin 2t * \sin 2t = \frac{1}{4}\int_0^t \sin(2t-2v)\sin(2v)\,dv$$

$$= \frac{1}{8}\int_0^t \Big(\underbrace{\cos(2t-4v)}_{=\cos(4v-2t)} - \cos 2t\Big)dt = \frac{1}{8}\left[\frac{1}{4}\sin(4v-2t) - v\cos 2t\right]\Big|_0^t$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 2t - t\cos 2t - \frac{1}{4}\sin(-2t)\right]$$

$$= \frac{1}{16}\left[\sin 2t - 2t\cos 2t\right]$$