1. (a) Verify that $\mathbf{u}_1 = \begin{bmatrix} 2+\sqrt{5} \\ 1 \end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$.

$$A\vec{u}_1 = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2+\sqrt{5} \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2\sqrt{5}+1 \\ 2+\sqrt{5}-2 \end{bmatrix} = \sqrt{5}\begin{bmatrix} 2+\sqrt{5} \\ 1 \end{bmatrix}$$ , i.e. $A\vec{u}_1 = \sqrt{5}\,\vec{u}_1$ ✓

(b) Determine all eigenvectors for the matrix $A$.

$|A - rI| = r^2 - 5 = 0$ so $r = \pm\sqrt{5}$ $\quad \vec{u}_1$ is an eigenvector for $r = \sqrt{5}$,

$r = -\sqrt{5}$

$$\begin{vmatrix} 2+\sqrt{5} & 1 \\ 1 & -2+\sqrt{5} \end{vmatrix} \vec{u}_2 = \vec{0}$$ , choose $\vec{u}_2 = \begin{bmatrix} 2-\sqrt{5} \\ 1 \end{bmatrix}$.

Note for $C \neq 0$, $C\vec{u}_1$ & $C\vec{u}_2$ are also eigenvectors.

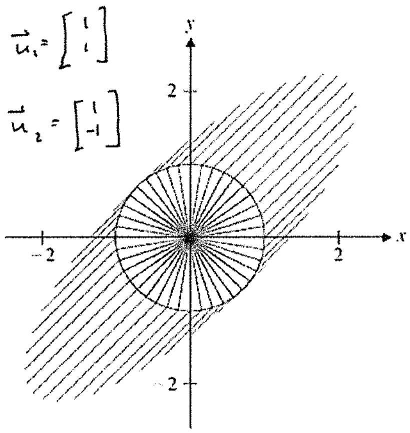
2. Is the Matrix $T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 5 \end{bmatrix}$ symmetric?

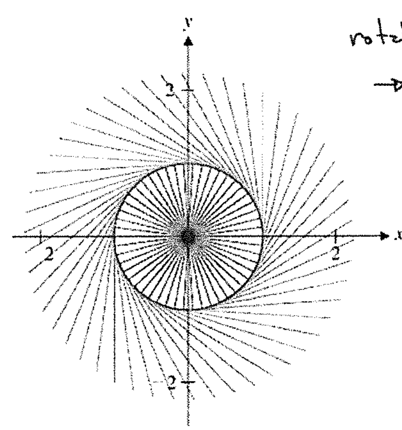
Nope!

3. The following describes multiplication by matrices $L$ and $R$ on real 2-vectors. Describe the eigenvalues/vectors for $L$ and $R$.

$r_1 = 2$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$r_2 = 0$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$




rotation → complex eigenvalues.

4. Solve the ivp:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 3 & 7 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{vmatrix} 3-r & 7 \\ 1 & -3-r \end{vmatrix} = r^2 - 16 = 0 \quad \text{so} \quad r = \pm 4$$

$r = 4$

$$\begin{bmatrix} -1 & 7 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \vec{0}$$

$r = -4$

$$\begin{bmatrix} 7 & 7 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}$$

so $\vec{x} = C_1 e^{4t}\begin{bmatrix} 7 \\ 1 \end{bmatrix} + C_2 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a general solution

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = C_1\begin{bmatrix} 7 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{so} \quad C_1 = 1,\ C_2 = -2$$

$$\vec{x} = e^{4t}\begin{bmatrix} 7 \\ 1 \end{bmatrix} - 2e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7e^{4t} - 2e^{-4t} \\ e^{4t} + 2e^{-4t} \end{bmatrix}$$

5. Given that $\mathbf{x}_p(t) = \begin{bmatrix} -t^{-1}/2 \\ t^{-1} \end{bmatrix}$ is a particular solution, describe the general solution to the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$, where $\mathbf{A} = \begin{bmatrix} 8 & -4 \\ 4 & -2 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} t^{-2}/2 \\ t^{-2} \end{bmatrix}$

$$\begin{vmatrix} 8-r & -4 \\ 4 & -2-r \end{vmatrix} = r^2 - 6r = r(r-6) = 0$$

$r = 0$

$$\begin{bmatrix} 8 & -4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{0}$$

$r = 6$

$$\begin{bmatrix} 2 & -4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{0}$$

$$\vec{x} = C_1 e^{6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -t^{-1}/2 \\ t^{-1} \end{bmatrix}$$