Math 274 In-Class **Worksheet 15a**

Sections: 9.8

In Section 9.8 we saw that if $\mathbf{X}(t)$ is a Fundamental Matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ then

$$e^{\mathbf{At}} = \mathbf{X}(t)\mathbf{X}^{-1}(0). \tag{1}$$

For any 2×2 matrix $\mathbf{A}$, the matrix exponential $e^{\mathbf{A}t}$ can be computed according to the table below.

Eigenvalues of $\mathbf{A}$	$e^{\mathbf{At}}$
r_1, r_2 real and distinct	$e^{r_1 t}\frac{1}{r_1 - r_2}(\mathbf{A} - r_2\mathbf{I}) - e^{r_2 t}\frac{1}{r_1 - r_2}(\mathbf{A} - r_1\mathbf{I})$
r real repeated twice	$e^{rt}\mathbf{I} + te^{rt}(\mathbf{A} - r\mathbf{I})$
$\alpha \pm i\beta$ complex conjugate pair	$e^{\alpha t}\cos(\beta t)\mathbf{I} + \frac{1}{\beta}e^{\alpha t}\sin(\beta t)(\mathbf{A} - \alpha\mathbf{I})$

1. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$.

   (a) Compute $e^{\mathbf{A}t}$ using (1) above.

   (b) Compute $e^{\mathbf{A}t}$ using the formula.

   (c) Note, $e^{\mathbf{A}t}$ is unique, so your solutions should be the same.

2. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$.

   (a) Compute $e^{\mathbf{A}t}$ using (1).

   (b) Compute $e^{\mathbf{A}t}$ using the formula.

   (c) Solve the initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

3. Consider the equation $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$.

   (a) In Section 9.5 #35 we saw that $\mathbf{A}$ had a repeated eigenvalue and found a solution making use of the idea of a *generalized eigenvector*. This was a non-trivial exercise.

   (b) Compute $e^{\mathbf{A}t}$ using the formula.