1. A rock contains two radioactive isotopes A and B, that belong to the same radioactive series; that is A decays into B, which then decays into stable atoms. Assume that the rate at which A decays into B is $40e^{-10t}$ kg/sec. Let $y(t)$ be the mass of B at time t. The rate of decay of B is proportional to the total mass of B present, i.e. $y' = -ky$.

   (a) Write a differential equation modeling the mass of B present at time t. Note, the amount of B is increasing as A decays and creates more B, but simultaneously decreasing as B decays. Assume the constant of proportionality in the decay of B is $k = 2$/sec.

$$\frac{dy}{dt} = \underbrace{40e^{-10t}}_{\text{Rate in}} - \underbrace{ky}_{\text{Rate out}} = 40e^{-10t} - 2y$$

   (b) Express your equation in standard linear form.

$$y' + 2y = 40e^{-10t}$$

   (c) Compute the integrating factor.

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

   (d) Find a general solution.

$$e^{2t}y' + 2e^{t}y = 40e^{-8t}$$

$$(e^{2t}y)' = 40e^{-8t}$$

$$e^{2t}y = \int 40e^{-8t}\,dt = -5e^{-8t} + C$$

$$y = Ce^{-2t} - 5e^{-8t}$$

   (e) If the mass of B is initially 20 kg, find the mass $y(t)$ of B as a function of t for $t \geq 0$.

$y(0) = 20$ so $C = 25$

$$y(t) = 25e^{-2t} - 5e^{-8t}$$

2. Consider the first order linear initial value problem

$$y' + \frac{y}{x-1} = \frac{5}{x^2-1}, \qquad y(0) = 1.$$

(a) Find an explicit solution to the initial value problem .

$\mu(x) = e^{\int \frac{dx}{x-1}} = e^{\ln|x-1|} = x-1$ choose

$(x-1)y' + y = \frac{5}{x+1}$

$[(x-1)y]' = \frac{5}{x+1}$

$(x-1)y = \int \frac{5}{x+1}\,dx = 5\ln|x+1| + C$

$y(x) = \frac{5\ln|x+1| + C}{x-1}$

$y(0) = 1 = \frac{C}{-1}$ so $C = -1$

so

$y(x) = \frac{5\ln(x+1) - 1}{x-1}$

(b) On what interval is your solution unique?

for $x \in (-1, 1)$

3. For the following initial value problems, are the given solutions unique.

(a) The initial value problem $y' = -\sqrt{y}$, $y(1) = 0$ has solution $y(t) = \frac{1}{4}(x-1)^2$.

$y \equiv 0$ is also a solution, so not unique.

$\left[\frac{\partial}{\partial y}(-\sqrt{y}) = -\frac{1}{2\sqrt{y}} \text{ is not continuous for } y = 0 \right]$

(b) The initial value problem $y' + 2xy = x$, $y(0) = 1$ has solution $y(x) = \frac{1}{2}(e^{-x^2} + 1)$.

$2x$ & x are continuous everywhere, so the solution

is unique on $(-\infty, \infty)$