Math 274 In-Class **Worksheet 3a**

Section: 2.6 and 3.2 (Kegs)

1. The magnetic field lines of a dipole (in the xy-plane) satisfy

$$\frac{dy}{dx} = \frac{3xy}{2x^2 - y^2}.$$

(a) Sketch a direction field for this differential equation.

find a computer!

(b) Is this equation homogeneous?

yes $\quad \frac{dy}{dx} = \frac{3\left(\frac{y}{x}\right)}{2 - \left(\frac{y}{x}\right)^2}$, Let $v = \frac{y}{x}$ so $xv = y$

$v + x\frac{dv}{dx} = \frac{dy}{dx}$

(c) Find a general solution of this ODE.

$$v + x\frac{dv}{dx} = \frac{3v}{2 - v^2}$$

$$x\frac{dv}{dx} = \frac{3v}{2-v^2} - \frac{v(2-v^2)}{2-v^2}$$

$$x\frac{dv}{dx} = \frac{v + v^3}{2-v^2}$$

so $$\int \frac{2-v^2}{v+v^3}\,dv = \int \frac{dx}{x}$$

A routine partial fractions exercise ~~shows~~ gives

$$\int \left(\frac{2}{v} - \frac{3v}{1+v^2}\right)dv = \ln|x| + C$$

so

$$2\ln|v| - \frac{3}{2}\ln|1+v^2| = \ln|x| + C$$

$$2\ln\left|\frac{y}{x}\right| - \frac{3}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| = \ln|x| + C$$

(d) Sketch some solution curves on the graphic you made in part (a).

2. Consider the differential equation

$$y' + \frac{y}{x} = x^2 y^2$$

(a) Find a general solution.

$y^{-2} y' + \frac{1}{x} y^{-1} = x^2$

Let $v = y^{-1}$

$v' = -y^{-2} y'$

so $v' - \frac{1}{x} v = -x^2$

$\mu(x) = e^{-\ln|x|} = \frac{1}{x}$

$\frac{1}{x} v' - \frac{1}{x^2} v = -x$

$\frac{1}{x} v = -\frac{x^2}{2} + C$

so $v = Cx - \frac{x^3}{2}$

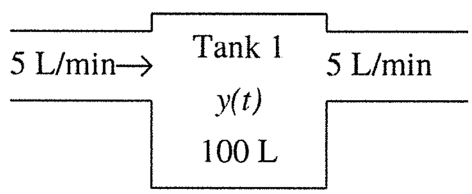
$\frac{1}{y} = Cx - \frac{x^3}{2}$

$y = \frac{2}{x(C - x^2)}$ or $y \equiv 0$

(b) Find a solution $y(x)$ such that $y(1) = 1$.

$y = \frac{2}{x(3 - x^2)}$

3. Consider a 100 L tank of pure water (imported from the Alps) into which a saline solution begins to flow at a constant rate of 5 L/min. The solution in the tank is well-mixed and flows out of the tank at 5 L/min. The concentration of the saline solution entering the tank is 0.5 kg/L.



(a) Letting $y(t)$ denote the mass of salt in the tank after t minutes, determine $y(t)$.

$\frac{dy}{dt} = \frac{5}{2} - 5\left(\frac{y}{100}\right)$, $y(0) = 0$

$\frac{dy}{dt} + \frac{1}{20} y = \frac{5}{2}$

$y e^{t/20} = \int \frac{5}{2} e^{t/20}\, dt$

$y = 50 + Ce^{-t/20}$

$y = 50 - 5e^{-t/20}$

(b) At what time will the concentration of salt in the tank reach 0.1 kg/L?

$y = 10$ when $50 - 50e^{-t/20} = 10$

$40 = 50 e^{-t/20}$

$e^{-t/20} = \frac{4}{5}$

$-t/20 = \ln(4/5)$

$t = 20 \ln(5/4)$

≈ 4.46