Math 274 In-Class **Worksheet 5p**

Sections: 4.4,4.5

Method of Undetermined Coefficients

To find a particular solution to

$$ay'' + by' + cy = P_m(t)e^{rt}$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s\left(A_m t^m + \cdots + A_1 t + A_0\right)e^{rt};$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root, take $s = 1$; and if r is a double root, take $s = 2$.

To find a particular solution to

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t$$

where $P_m(t)$ and $Q_n(t)$ are polynomials of degree m and n, respectively, use the form

$$y_p(t) = t^s\left(A_k t^k + \cdots + A_1 t + A_0\right)e^{\alpha t}\cos\beta t + t^s\left(B_k t^k + \cdots + B_1 t + B_0\right)e^{\alpha t}\sin\beta t;$$

where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if so take $s = 1$.

1. Find the appropriate form using the Method of Undetermined Coefficients for a particular solution to the following. **Do not** solve for the unknown constants.

   (a) $y'' - 4y' + 4y = t\sin 3t$ $\quad y_p = (At+B)\sin 3t + (Ct+D)\cos 3t$

   $r^2 - 4r + 4 = 0$

   $(r-2)^2 = 0$

   (b) $y'' - 4y' + 4y = te^t + 3e^{4t}$ $\quad y_p = (At+B)e^t + Ce^{4t}$

   (c) $y'' - y' - 6y = e^{3t} + 7t^2$ $\quad y_p = Ate^{3t} + Bt^2 + Ct + D$

   $r^2 - r - 6 = 0$

   $(r-3)(r+2) = 0$

   $\rightarrow r = 3$

   (d) $y'' + 9y = \sin 3t$ $\quad y_p = At\sin 3t + Bt\cos 3t$

   $r = \pm 3i$

   (e) $y'' + 9y = e^{2t}\cos 3t$ $\quad y_p = Ae^{2t}\cos 3t + Be^{2t}\sin 3t$

   $r = \pm 3i$

STOP! Don't move on until you know these are correct.

2. Find a general solution for the following.

(a) $y'' + y = 4$

$r^2 + 1 = 0$

$r = \cancel{\pm 1}$

$r = \pm i$

$y_p = A$

$y_p' = y_p'' = 0$

so $A = 4$

$\cancel{y = C_1 e^t + C_2 e^{-t} + 4}$

$y = C_1 \cos t + C_2 \sin t + 4$

(b) $y'' + y = 4t$

$y_p = At + B$

$y_p' = A, \ y_p'' = 0$

so $0 + At + B = 4t$

$\cancel{y = C_1 e^t + C_2 e^{-t} + 4t}$

$y = C_1 \cos t + C_2 \sin t + 4t$

(c) $y'' = 4$

$r^2 = 0$

$y_p = At^2$

$y_p' = 2At$

$y_p'' = 2A$ so $A = 2$

$y = C_1 + C_2 t + 2t^2$

(d) $y'' + 6y' + 8y = 16 + 6e^{-2t}$

$r^2 + 6r + 8 = 0$

$(r+2)(r+4) = 0$

$r = -2, \ r = -4$

$y_p = A + Bte^{-2t}$

$y_p' = Be^{-2t} - 2Bte^{-2t}$

$y_p'' = -4Be^{-2t} + 4Bte^{-2t}$

$\underline{t^0}$: $8A = 16$ so $A = 2$

$\underline{e^{-2t}}$: $2B = 6$ so $B = 3$

Substituting gives

$$e^{-2t}\left[-4B + 4Bt\right] + e^{-2t}\left[6B - 12Bt\right] + 8A + e^{-2t}\left[8Bt\right] = 16 + 6e^{-2t}$$

$$y = C_1 e^{-2t} + C_2 e^{-4t} + 2 + 3te^{-2t}$$

(e) $y'' + 6y' + 8y = \cos 2t$

$y_p = A \cos 2t + B \sin 2t$

$y_p' = -2A \sin 2t + 2B \cos 2t$

$y_p'' = -4A \cos 2t - 4B \sin 2t$

$$y = C_1 e^{-2t} + C_2 e^{-4t} + \frac{1}{40} \cos 2t + \frac{3}{40} \sin 2t$$

$y_p'' + 6y_p' + 8y_p = \cancel{\cos 2t \left[A + 12B - 32A\right] +}$

$$= \cos 2t\left[-4A + 12B + 8A\right] + \sin 2t\left[-4B - 12A + 8B\right] = \cos 2t$$

$$\left.\begin{array}{l} \cos t: \ 4A + 12B = 1 \\ \sin t: \ 4B - 12A = 0 \end{array}\right\} \quad A = \frac{1}{40}, \ B = \frac{3}{40}$$