Math 274 In-Class **Worksheet 6a**
Sections: 4.6

Variation of Parameters

If y_1 and y_2 are linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then a particular solution to $y'' + p(t)y' + q(t)y = g(t)$ is given by

$$y_p(t) = y_1(t) \int \frac{-g(t)y_2(t)}{W[y_1, y_2](t)}\,dt + y_2(t) \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,dt.$$

1. We are interested in solving the initial value problem

$$y'' + 9y = \csc 3t, \qquad y(\pi/6) = y'(\pi/6) = 0. \tag{1}$$

(a) Find a general solution to the associated homogeneous equation, i.e $y'' + 9y = 0$.

$r^2 + 9 = 0$ so $r = \pm 3i$

$y = C_1 \cos 3t + C_2 \sin 3t$

(b) Find a particular solution to $y'' + 9y = \csc 3t$.

$$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 3t + 3\sin^2 3t = 3$$

$$y = \cos 3t \int -\frac{\csc 3t \cdot \sin 3t}{3} + \sin 3t \int \frac{\csc 3t \ \cos 3t}{3}$$

$$= -\frac{t}{3}\cos 3t + \frac{\sin 3t}{3}\int \frac{\cos 3t}{\sin 3t}\,dt = -\frac{t}{3}\cos 3t + \frac{1}{9}\sin 3t \ln|\sin 3t|$$

(c) Find a general solution to $y'' + 9y = \csc 3t$.

$$y = C_1 \cos 3t + C_2 \sin 3t - \frac{t}{3}\cos 3t + \frac{1}{9}\sin 3t \ln|\sin 3t|$$

(d) Find the solution to the initial value problem (1).

$y(\pi/6) = 0 \Rightarrow C_2 = 0$

$$y' = -3C_1 \sin 3t - \frac{1}{3}\cos 3t + t\sin 3t + \frac{1}{3}\cos 3t \ln|\sin 3t| + \frac{1}{3}\sin 3t \cdot \frac{\cos 3t}{\sin 3t}$$

$y'(\frac{\pi}{6}) = -3C_1 + \frac{\pi}{6} = 0$ so $C_1 = +\frac{\pi}{18}$ so

$$y = \frac{\pi}{18}\cos 3t - \frac{t}{3}\cos 3t + \frac{1}{9}\sin 3t \ln|\sin 3t|$$

2. Consider the equation

$$y'' + 9y = 9\sin 3t.$$

(a) Find a particular solution using Variation of Parameters. You may find the following trigonometric identities useful.

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\sin 2x = 2\sin x\cos x$

$y_1 = \cos 3t \quad , \quad y_2 = \sin 3t \qquad W = 3$

$$y_p = \cos 3t \int \frac{-9\sin^2 3t}{3}\,dt + \sin 3t \int \frac{9\sin 3t\cos 3t}{3}\,dt$$

Note: $\sin^2 3t = \frac{1}{2}(1-\cos 6t)$

$$= \cos 3t\left(-\frac{3}{2}\right)\left(t - \frac{1}{6}\sin 6t\right) + \frac{1}{2}\sin 3t \cdot \sin^2 3t$$

Note: $\frac{1}{6}\sin 6t = \frac{1}{3}\sin 3t\cos 3t$

$$= -\frac{3}{2}t\cos 3t + \frac{1}{2}\sin 3t\cos^2 3t + \frac{1}{2}\sin 3t \cdot \sin^2 3t$$

$$= -\frac{3}{2}t\cos 3t + \underbrace{\frac{1}{2}\sin 3t}$$

Part of the homogeneous, so not needed, but fine.

(b) Find a particular solution using the Method of Undetermined Coefficients.

$y_p = At\cos 3t + Bt\sin 3t$

$y_p' = (A + 3Bt)\cos 3t + (B - 3At)\sin 3t$

$y_p'' = (6B - 9At)\cos 3t + (-6A - 9Bt)\sin 3t$

$y_p'' + 9y_p = 6B\cos 3t - 6A\sin 3t = 9\sin 3t$

so $B = 0$, $A = -3/2$

$y_p = -\frac{3}{2}t\cos 3t$