Math 274 In-Class **Worksheet 6b**

Section: 4.6 and 4.7

1. Let $t > 0$, consider the equation

$$t^2y'' - 4ty' + 4y = t^5 \sin t.$$

(a) Find a general solution to the corresponding homogeneous equation.

Try $y = t^r$

$y' = r t^{r-1}$

$y'' = r(r-1) t^{r-2}$

Substituting gives

$t^2 (r^2 - r) t^{r-2} - 4 t r t^{r-1} + 4 t^r = 0$

$t^r \left[r^2 - 5r + 4 \right] = 0$

$(r-1)(r-4) = 0$

$r_1 = 1 \quad , \quad r_2 = 4$

so $y = C_1 t + C_2 t^4$ is a general solution.

We will need the Wronskian below,

so

$$W[t, t^4] = \begin{vmatrix} t & t^4 \\ 1 & 4t^3 \end{vmatrix} = 4t^4 - t^4 = 3t^4$$

(b) Find a particular solution to the inhomogeneous equation above.

Putting in standard form gives $y'' - \frac{4}{t} y' + \frac{4}{t^2} y = t^3 \sin t$

$$y_p = t \int \frac{(-t^3 \sin t)\, t^4}{3t^4} dt + t^4 \int \frac{(t^3 \sin t)\, t}{3t^4} dt = \frac{t^4}{3}(-\cos t) - \frac{t}{3} \int t^3 \sin t \, dt$$

Applying integration by parts (three times) to the last integral gives

$$y_p = -\frac{t^4}{3}(\cos t) - \frac{t}{3}\left[-t^3 \cos t + 3t^2 \sin t + 6t \cos t - 6 \sin t \right]$$

$$= -t^3 \sin t - 2t^2 \cos t + 2t \sin t$$

2. Use the Wronskian to verify that the second solution found via Reduction of Order is linearly independent from the first, i.e., ~~vefiry~~ Verify that if $y_1(t) \not\equiv 0$, then $y_1(t)$ and

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2} dt$$

are linearly independent.

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_1 \int \frac{e^{-\int p}}{y_1^2} \\ y_1' & y_1' \int \frac{e^{-\int p}}{y_1^2} + y_1 \cdot \frac{e^{-\int p}}{y_1^2} \end{vmatrix}$$

$$= e^{-\int p(t)dt} \neq 0 \quad \text{for all } t.$$

3. For $x > 0$, find a second linearly independent solution to

$$xy'' - y' + (1-x)y = 0,$$

provided that $y_1 = e^x$ is a solution.

Putting in standard form we have $y'' - \frac{1}{x}y' + \left(\frac{1-x}{x}\right)y = 0$.

$$y_2 = e^x \int \frac{e^{-\int -\frac{1}{x}dx}}{e^{2x}} dx = e^x \int x e^{-2x} dx$$

$u = x \qquad dv = e^{-2x}dx$

$du = dx \qquad v = -\frac{1}{2}e^{-2x}$

$$= e^x \left[-\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x} dx \right]$$

$$= e^x \left(-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} \right)$$

$$= -\frac{x}{2}e^{-x} - \frac{1}{4}e^{-x} = e^{-x}\left(-\frac{x}{2} - \frac{1}{4}\right) = -\frac{e^{-x}}{4}(2x+1) = \ldots$$