Math 274 In-Class **Worksheet 7a**

Sections: 4.9, and more

1. Consider mass-spring systems characterized by the following initial value problems. Characterize each as undamped, underdamped, critically damped, or overdamped. Solve each initial value problem and sketch the solution.

   (a) $x'' + 2x' + x = 0, \qquad x(0) = x'(0) = 1$

   $r^2 + 2r + 1 = 0$

   $(r+1)^2 = 0$

   $r = -1$, repeated

   $x(t) = C_1 e^{-t} + C_2 t e^{-t}$

   $x(0) = 1 \Rightarrow C_1 = 1$

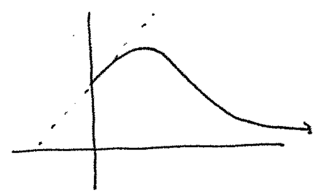
   $x'(t) = -e^{-t} + C_2 e^{-t} - C_2 t e^{-t}$

   $x'(0) = -1 + C_2 = 1 \Rightarrow C_2 = 2$

   $x(t) = e^{-t} + 2te^{-t}$

   $= e^{-t}(1 + 2t)$

   Critically Damped

   (b) $x'' + 3x' + 2x = 0, \qquad x(0) = x'(0) = 1$

   $r^2 + 3r + 2 = 0$

   $(r+2)(r+1) = 0$

   $r = -1, -2$

   $x(t) = C_1 e^{-t} + C_2 e^{-2t}$

   $x(0) = C_1 + C_2 = 1$

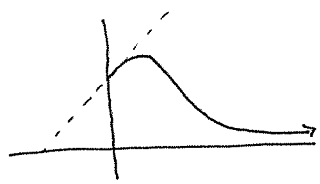
   $x'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$

   $x'(0) = -C_1 - 2C_2 = 1$

   so $C_1 = 3$, $C_2 = -2$

   $x(t) = 3e^{-t} - 2e^{-2t}$

   Over damped

   (c) $x'' + 4x' + 5x = 0, \qquad x(0) = x'(0) = 1$

   $r^2 + 4r + 5 = 0$

   $(r+2)^2 + 1 = 0$

   $r = -2 \pm i$

   $x(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$

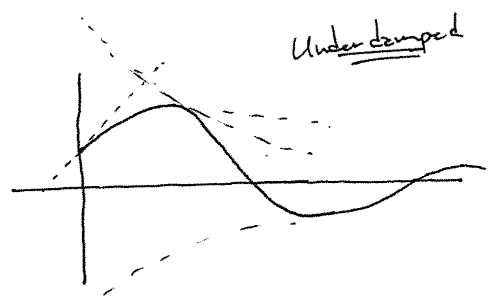
   $x(0) = C_1 = 1$

   $x'(t) = -2e^{-2t}\cos t - e^{-2t}\sin t - 2e^{-2t}\sin t + C_2 e^{-2t}\cos t$

   $x'(0) = -2 + C_2 = 1 \Rightarrow C_2 = 3$

   $x(t) = e^{-2t}(\cos t + 3\sin t)$

   Underdamped

   (d) $x'' + 4x = 0, \qquad x(0) = x'(0) = 1$

   $r^2 + 4 = 0$

   $r = \pm 2i$

   $x(t) = C_1 \cos 2t + C_2 \sin 2t$

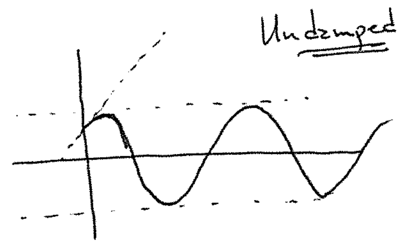
   $x(0) = C_1 = 1$

   $x'(t) = -2\sin 2t + 2C_2 \cos 2t$

   $x'(0) = 2C_2 = 1 \Rightarrow C_2 = \frac{1}{2}$

   $x(t) = \cos 2t + \frac{1}{2}\sin 2t$

   Undamped

2. Consider a mass-spring system that has an external forcing function that turns off at $t = \pi/2$ modeled by the following initial value problem

$$y'' + 2y' + 2y = \begin{cases} 1, & t < \pi/2 \\ 0, & \pi/2 < t \end{cases}, \qquad y(0) = y'(0) = 0.$$

Find a solution to the initial value problem. Note, your solution will be piecewise defined.

We do this in two parts, first, for $0 \le t < \frac{\pi}{2}$

$y'' + 2y' + 2y = 1$

has solution $y = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + \frac{1}{2}$

$y(0) = 0 = C_1 + \frac{1}{2}$ so $C_1 = -\frac{1}{2}$

$y' = +\frac{1}{2}(e^{-t})\cos t + \frac{1}{2}e^{-t}\sin t - C_2 e^{-t}\sin t + C_2 e^{-t}\cos t$ (*)

$y'(0) = \frac{1}{2} + C_2 = 0$ so $C_2 = -\frac{1}{2}$

For $t \in [0, \pi/2)$ the solution is

$$y = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$$

Evaluating this at $t = \pi/2$ will give new initial data for the next piece

$y(\pi/2) = \frac{1}{2} - \frac{1}{2}e^{-\pi/2} = \frac{1}{2}(1 - e^{-\pi/2})$

$y'(t) = \frac{1}{2}(e^{-t})(\cos t + \sin t) - \frac{1}{2}e^{-t}(-\sin t + \cos t) = e^{-t}\sin t$

$y'(\pi/2) = e^{-\pi/2}$

For $t > \pi/2$ we have

$y'' + 2y' + 2y = 0$ which has solution $y = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t$

$y(\pi/2) = \frac{1}{2}(1 - e^{-\pi/2}) \Rightarrow C_2 e^{-\pi/2} = \frac{1}{2}(1 - e^{-\pi/2})$ or $C_2 = \frac{1}{2}(e^{\pi/2} - 1)$

$y'(t) = -C_1 e^{-t}\cos t - C_1 e^{-t}\sin t - \frac{1}{2}(e^{\pi/2} - 1)e^{-t}\sin t + \frac{1}{2}(e^{\pi/2} - 1)e^{-t}\cos t$

$y'(\pi/2) = -C_1 e^{-\pi/2} - \frac{1}{2}(e^{\pi/2} - 1)e^{-\pi/2} = e^{-\pi/2}$

so $C_1 = -\frac{1}{2}(e^{\pi/2} - 1)$

so $y = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t) & t < \frac{\pi}{2} \\ \frac{1}{2}(e^{\pi/2} - 1)(\sin t - \cos t)e^{-t} & t > \frac{\pi}{2} \end{cases}$