Math 274 In-Class **Worksheet 9p**

Section: 7.6

1. Express the following functions using step functions and determine their Laplace transforms.

(a) $f(t) = \begin{cases} 0, & t < 1 \\ 1, & 1 < t < 2 \\ 2, & 2 < t \end{cases}$ $= 0 + u(t-1)(1-0) + u(t-2)(2-1)$

$= u(t-1) + u(t-2)$

so $F(s) = \dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s}$

(b) $g(t) = \begin{cases} t, & t < \pi \\ e^{3t}\cos 2t, & \pi < t \end{cases}$ $= t + u(t-\pi)(e^{3t}\cos 2t - t)$

$G(s) = \dfrac{1}{s^2} + e^{-\pi s}\mathcal{L}\{e^{(3t+3\pi)}\cos(2t+2\pi) - (t+\pi)\}$

Note: $\cos(2t+2\pi) = \cos 2t$

$= \dfrac{1}{s^2} + e^{-\pi s}\left[e^{3\pi}\cdot\dfrac{s-3}{(s-3)^2+2^2} - \dfrac{1}{s^2} - \dfrac{\pi}{s}\right]$

2. Determine the inverse Laplace transform for the following.

(a) $H(s) = \dfrac{1}{s} + \dfrac{e^{-2s}}{s^2} + \dfrac{e^{-4s}}{s^3}$

$h(t) = 1 + u(t-2)(t-2) + \frac{1}{2}u(t-4)(t-4)^2$

(b) $K(s) = \dfrac{8e^{4-2s}}{(s-2)(s^2+4)}$ $= e^4 e^{-2s}\left[\dfrac{8}{(s-2)(s^2+4)}\right]$

$\dfrac{8}{(s-2)(s^2+4)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+4}$

$K(s) = e^4 e^{-2s}\left[\dfrac{1}{s-2} - \dfrac{s}{s^2+4} - \dfrac{2}{s^2+4}\right]$

$8 = A(s^2+4) + (Bs+C)(s-2)$

$k(t) = e^4 u(t-2)\left[e^{2(t-2)} - \cos(2t-4) - \sin(2t-4)\right]$

$s = 2$: $A = 1$

s^2 : $0 = A + B$ so $B = -1$

s^0 : $8 = 4A - 2C$ so $C = -2$

3. Applying the Laplace transform to the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, y'(0) = 0$$

with

$$f(t) = \begin{cases} 10e^t, & t < 1 \\ -8e^{2t}, & 1 < 2 < t \\ 0, & 2 < t \end{cases}$$

gives

$$Y(s) = \frac{10}{(s-1)(s^2+4)} + e^{-s}\left[\frac{-8e^2}{(s-2)(s^2+4)} - \frac{10e}{(s-1)(s^2+4)}\right] + e^{-2s}\left[\frac{8e^4}{(s-2)(s^2+4)}\right].$$

Determine $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, the solution to the given initial value problem.

You should reference the partial fraction decomposition in 2(b). Additionally, a routine exercise shows $\frac{10}{(s-1)(s^2+4)} = \frac{2}{s-1} - \frac{2s+2}{s^2+4}$. From 2(b) we know $\frac{8}{(s-2)(s^2+4)} = \frac{1}{s-2} - \frac{s+2}{s^2+4}$

so

$$Y(s) = \left[\frac{2}{s-1} - \frac{2s+2}{s^2+4}\right] + e^{-s}\left[-e^2\left(\frac{1}{s-2} - \frac{s+2}{s^2+4}\right) - e\left(\frac{2}{s-1} - \frac{2s+2}{s^2+4}\right)\right] + e^{-2s}e^4\left[\frac{1}{s-2} - \frac{s+2}{s^2+4}\right]$$

giving

$$\begin{aligned} y(t) = {} & 2e^t - 2\cos 2t - \sin 2t \\ & + u(t-1)\Big[-e^{+2}\left(e^{2(t-1)} - \cos(2t-2) - \sin(2t-2)\right) \\ & \qquad - e\left(2e^{t-1} - 2\cos(2t-2) - \sin(2t-2)\right)\Big] \\ & + u(t-2)\Big[e^4\left(e^{2(t-2)} - \cos(2t-4) - \sin(2t-4)\right)\Big] \end{aligned}$$

Exercise: Using your favorite piece of technology, generate a picture of the graph of y. It should look something like this.