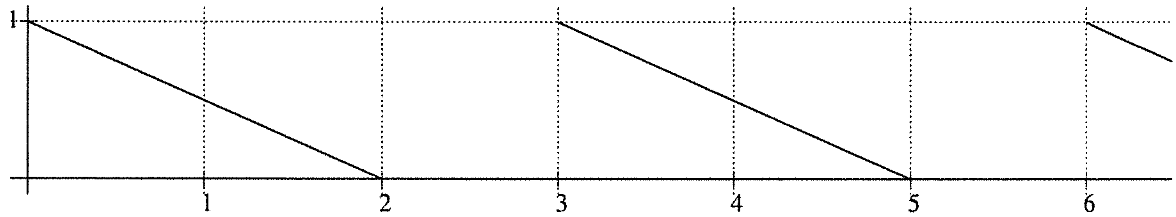
Math 274 In-Class **Worksheet 10a**

Sections: 7.7, 7.8

1. Compute the Laplace transform of the periodic function $f(t)$ given by the graph below.



$$f_3(t) = \begin{cases} 1 - t/2 & 0 < t < 2 \\ 0 & \text{otherwise} \end{cases} = 1 - t/2 - u(t-2)\left(1 - t/2\right)$$

$= -t/2$

$$F_3(s) = \frac{1}{s} - \frac{1}{2s^2} - e^{-2s}\mathcal{L}\left\{1 - \frac{t+2}{2}\right\}$$

$$= \frac{1}{s} - \frac{1}{2s^2} - e^{-2s}\left(-\frac{1}{2s^2}\right)$$

$$F(s) = \frac{\frac{1}{s} - \frac{1}{2s^2} + \frac{e^{-2s}}{2s^2}}{1 - e^{-3s}} = \frac{2s - 1 + e^{-2s}}{2s^2\left(1 - e^{-3s}\right)}$$

2. Let $f(t) = e^t$, $0 < t < 1$ and extend periodically to a function defined on the positive reals. Apply the Laplace transform and solve for $Y(s)$ in the initial value problem

$$y'' - y = f(t), \qquad y(0) = y'(0) = 0.$$

$$f_1(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases} = e^t - u(t-1)e^t$$

$$F_1(s) = \frac{1}{s-1} - e^{-s}\mathcal{L}\{e^{t+1}\} = \frac{1}{s-1} - e^{-s}\frac{e}{s-1} = \frac{1 - e^{1-s}}{(s-1)(1-e^{-s})}$$

$$s^2 Y - Y = \frac{1 - e^{1-s}}{(s-1)(1-e^{-s})}$$

$$Y = \frac{1 - e^{1-s}}{(s^2-1)(s-1)\underset{(1-e^{-s})}{}} = \frac{1 - e^{1-s}}{(s-1)^2(s+1)(1-e^{-s})}$$

3. Compute the convolution $f(t) * g(t)$ for the following:

(a) $f(1) = 1,\ g(t) = t^2$;
(b) $f(t) = t,\ g(t) = t$;
(c) $f(t) = 1 * 1,\ g(t) = t$;
(d) $f(t) = t,\ g(t) = e^t$.

$$f * g = \int_0^t f(t-v)\,g(v)\,dv$$

$$1 * t^2 = \int_0^t v^2\,dv = \left.\frac{v^3}{3}\right|_0^t = \frac{t^3}{3}$$

$$t * t = \int_0^t (t-v)v\,dv = \left.\frac{tv^2}{2} - \frac{v^3}{3}\right|_0^t = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6}$$

$$1 * 1 = \int_0^t 1\,dv = v\Big|_0^t = t \quad \text{so} \quad (1*1) * t = t * t = \frac{t^3}{6}$$

$$t * e^t = \int_0^t (t-v)e^v\,dv = (t-v)e^v\Big|_0^t + \int_0^t e^v\,dv = 0 - t + e^v\Big|_0^t$$

$$U = t-v \qquad dV = e^v dv$$
$$dU = -dv \qquad V = e^v$$

$$= e^t - t - 1$$

4. Verify that $\mathscr{L}\{f * g\}(s) = F(s)G(s)$ for cases (a) and (d) from the preceding problem.

$$\mathscr{L}\{1 * t^2\} = \mathscr{L}\left\{\frac{t^3}{3}\right\} = \frac{1}{3}\cdot\frac{3!}{s^4} = \frac{2}{s^4} \quad \text{\& } \mathscr{L}\{1\}\,\mathscr{L}\{t^2\} = \frac{1}{s}\cdot\frac{2}{s^3} = \frac{2}{s^4}\ \checkmark$$

$$\mathscr{L}\{t * e^t\} = \mathscr{L}\{e^t - t - 1\} = \frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{s} = \frac{s^2 - (s-1) - s(s-1)}{s^2(s-1)} = \frac{1}{s^2(s-1)}$$

$$\text{\& } \mathscr{L}\{t\}\,\mathscr{L}\{e^t\} = \frac{1}{s^2}\cdot\frac{1}{s-1}\ \checkmark$$