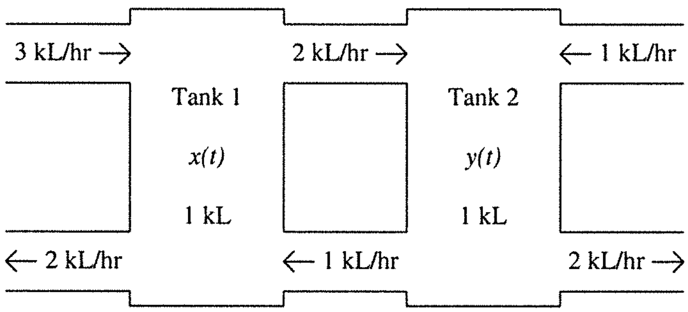
1. Tank 1 initially contains 6 kg of salt dissolved into 1 kL of water. Tank 2 initially contains 9 kg of salt dissolved into 1 kL of water. Both tanks are well mixed. Pure water is flowing into each tank at the rate specified in the figure. Similarly, the figure shows the rate the mixtures are flowing between each tank and being drained. Let $x(t)$ be the amount of salt in tank 1 in kg, and $y(t)$ be the amount of salt in tank 2 in kg.



Find the amount of salt in each tank as a function of time.

(1) $x' = -4x + y \qquad x(0) = 6$

$y' = 2x - 3y \qquad y(0) = 9$

$2x = y' + 3y \qquad (2)$

$2x' = y'' + 3y' \qquad (3)$

Multiply (1) by 2 & substitute (2) & (3) gives

$y'' + 3y' = -4(y' + 3y) + 2y$

$y(0) = 9 \qquad y'(0) = 2x(0) - 3y(0) = -15$

$y'' + 7y' + 10y = 0$

$r^2 + 7r + 10 = 0$

$r = -2, -5$

$y = C_1 e^{-2t} + C_2 e^{-5t}$ so $C_1 + C_2 = 9$

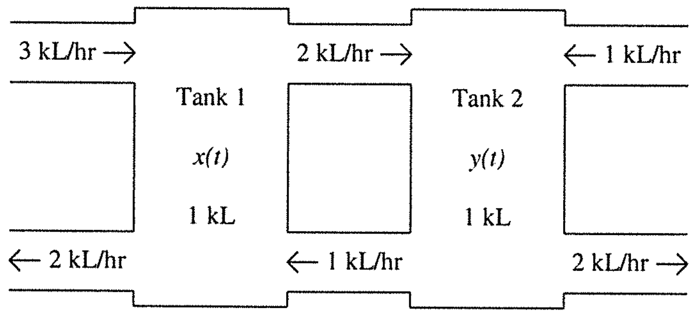
$y' = -2C_1 e^{-2t} - 5C_2 e^{-5t}$ so $-2C_1 - 5C_2 = -15$

$C_1 = 10, \quad C_2 = -1$ solves, so

$y = 10e^{-2t} - e^{-5t}$

$x = 5e^{-2t} + e^{-5t}$

2. Tank 1 initially contains 6 kg of salt dissolved into 1 kL of water. Tank 2 initially contains 9 kg of salt dissolved into 1 kL of water. Both tanks are well mixed. Pure water is flowing into each tank at the rate specified in the figure. Similarly, the figure shows the rate the mixtures are flowing between each tank and being drained. Let $x(t)$ be the amount of salt in tank 1 in kg, and $y(t)$ be the amount of salt in tank 2 in kg.



After two hours the night janitor accidentally bumps a switch causing the input into tank 1 to be a mixture containing 10 kg of salt per kL instead of pure water. Find the amount of salt in each tank for $t \in (0,2)$ and for $t \in (2,\infty)$.

$x' = -4x + y + 30u(t-2)$ $\quad x(0) = 6$

$y' = 2x - 3y$ $\quad y(0) = 9$

As before, substitute to find

$y'' + 7y' + 10y = 60u(t-2)$

Taking the Laplace transform, simplifying, and doing partial fractions gives

$$Y = e^{-2s}\left[\frac{6}{s} + \frac{4}{s+5} - \frac{10}{s+2}\right] + \left(\frac{10}{s+2} - \frac{1}{s+5}\right)$$

$$y(t) = u(t-2)\left[3 + 2e^{10}e^{-5t} - 5e^{4}e^{-2t}\right] + 10e^{-2t} - e^{-5t}$$

so $$y(t) = \begin{cases} 10e^{-2t} - e^{-5t} & t < 2 \\ 3 + (2e^{10} - 1)e^{-5t} + (10 - 5e^{4})e^{-2t} & t > 2 \end{cases}$$

$$x(t) = \begin{cases} 5e^{-2t} + e^{-5t} & t < 2 \\ \frac{1}{2}\left[2(2e^{10} - 1)e^{-5t} + (10 - 5e^{4})e^{-2t} + 9\right] & t > 2 \end{cases}$$