Math 274 In-Class **Worksheet 14**

Section: 9.4

1. (a) Show that the following vectors are linearly dependent

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \\ -4 \end{bmatrix} \right\}$$

typo on original

$$3\begin{bmatrix} 1 \\ 4 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 7 \\ 10 \\ -4 \\ -1 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 1 \\ 5 \\ -4 \end{bmatrix} = \vec{0}$$

(b) Are the following vectors linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \right\}$$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} + \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \vec{0}$, so they are linearly <u>dependent</u>.

(c) Are the following vectors linearly independent on $(-\infty, \infty)$?

$$\left\{ \begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}, \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} \right\}$$

yes.

$\begin{vmatrix} te^{-t} & e^{-t} \\ e^{-t} & e^{-t} \end{vmatrix} = (t-1)e^{-2t} \neq 0$ for $t \neq 1$,

so they are Linearly Independent.

(Lin. Dep $\Rightarrow$ $W[\vec{x}_1, \vec{x}_2] = 0$ for all t)

2. On which interval does the following initial value problem have a unique solution?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \tan t & 0 \\ t & \ln t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 1/t \end{bmatrix}, \quad \begin{bmatrix} x_1(\pi/4) \\ x_2(\pi/4) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$t \in (0, \frac{\pi}{2})$

3. (a) Let $\mathbf{X}(t)$ be a fundamental matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Show that $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ is a solution to the ivp given by the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

typo on original.

$\vec{x}(t) = \underline{X}(t)\vec{c}$ is a general solution.

Substituting the initial data gives

$\vec{x}_0 = \vec{x}(t_0) = \underline{X}(t_0)\vec{c}$. Since $\underline{X}$ is a fundamental matrix, it is invertible, so

$\vec{c} = \underline{X}^{-1}(t_0)\vec{x}_0$ & the solution is then $\vec{x}(t) = \underline{X}(t)\underline{X}^{-1}(t_0)\vec{x}_0$

(b) Verify $\mathbf{X}(t)$ is a fundamental matrix and solve the ivp.

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}$$

$$\underline{X}' = \begin{bmatrix} -e^{-t} & 5e^{5t} \\ e^{-t} & 5e^{5t} \end{bmatrix}$$

$$A\underline{X} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix} = \begin{bmatrix} -e^{-t} & 5e^{5t} \\ e^{-t} & 5e^{5t} \end{bmatrix}$$

so $\underline{X}' = A\underline{X}$

$|\underline{X}| = e^{4t} + e^{4t} \neq 0$, so Lin. Ind.

$\underline{X}(0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\underline{X}^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

$\underline{X}^{-1}(0)\vec{x}_0 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so $\vec{x} = \underline{X}(t)\underline{X}^{-1}(0)\vec{x}_0 = \begin{bmatrix} 2e^{-t} + e^{5t} \\ -2e^{-t} + e^{5t} \end{bmatrix}$