Math 274 In-Class **Worksheet 11a**

Section: 7.9

1. Consider the mass-spring system given by the initial value problem

$$x'' + 2x' + 5x = 0, \qquad x(0) = 0, x'(0) = 2. \qquad (1)$$

(a) Find the solution to (1).

$$s^2 X - 2 + 2sX + 5X = 0$$

$$X = \frac{2}{(s+1)^2 + 2^2}$$

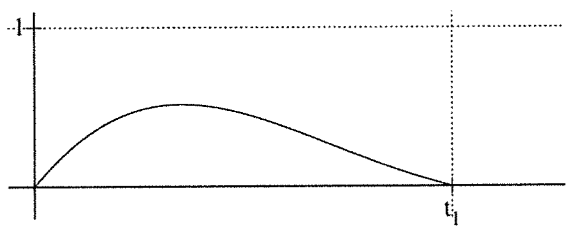
$$x = e^{-t} \sin 2t \qquad \text{so} \quad x' = -e^{-t}\sin 2t + 2e^{-t}\cos 2t$$

$$x'(\pi/2) = -2e^{-\pi/2}$$

(b) Find the magnitude of the impulse needed to stop the motion of the system when it first returns to equilibrium at time t_1, i.e., find M so that the solution to the symbolic initial value problem

$$x'' + 2x' + 5x = M\delta(t - \pi/2), \qquad x(0) = 0, x'(0) = 2$$

has the following graph.



You may find the following useful.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Since $x'(\frac{\pi}{2}) = -2e^{-\pi/2}$ $\quad M = 2e^{-\pi/2}$

— OR —

$$X(s^2 + 2s + 5) = 2 + Me^{-\pi/2\, s}$$

$$x(t) = e^{-t}\sin 2t + \frac{M}{2}u(t - \tfrac{\pi}{2})e^{-(t - \frac{\pi}{2})}\sin(2t - \pi)$$

$$= e^{-t}\sin 2t\underbrace{\left[1 - \frac{M}{2}e^{\pi/2}u(t - \tfrac{\pi}{2})\right]}$$

we want this to be 0 for $t > \frac{\pi}{2}$,

so $M = 2e^{-\pi/2}$

2. Find the solution to the symbolic initial value problem

$$y'' + 2\pi y' + 5\pi^2 y = 4\pi\delta(t-1), \qquad x(0) = 0, x'(0) = 2\pi.$$

$$Y\left(s^2 + 2\pi s + 5\pi^2\right) = 4\pi e^{-s} + 2\pi$$

$$Y = 2e^{-s} \cdot \frac{2\pi}{(s+\pi)^2 + (2\pi)^2} + \frac{2\pi}{(s+\pi)^2 + (2\pi)^2}$$

$$y = 2u(t-1)\, e^{-\pi(t-1)} \sin\left(2\pi(t-1)\right) + e^{-\pi t} \sin(2\pi t)$$

$$= \begin{cases} e^{-\pi t} \sin 2\pi t & t < 1 \\ (2e^{\pi}+1) e^{-\pi t} \sin 2\pi t & t > 1 \end{cases}$$

3. Use scratch paper, and your remaining time to investigate the following.

(a) For $n > 0$, consider the initial value problem

$$y'' + y = n\left(1 - u(t - 1/n)\right), \qquad y(0) = y'(0) = 0.$$

Find the solution, $y_n(t)$, and express it as a piecewise defined function that depends on n.

(b) Evaluate

$$\lim_{n\to\infty} y_n(t).$$

(c) Solve the initial value problem

$$y'' + y = \delta(t), \qquad y(0) = y'(0) = 0.$$

(d) Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 0, y'(0) = 1.$$

* This example is in the textbook.

(e) What do you notice about the solutions to (b), (c), and (d)? Is it what you expected?