Math 172 | Exam 3 | Name: ________________
30 May 2013 | Show Appropriate Work | Point Values in [Brackets]

1. [5] Express the following in its partial fraction form. You do not need to solve for the coefficients.

$$\frac{1}{(x+1)^3(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+1}$$

2. Integrate.

(a) [10] $\int \frac{5x^2+x+3}{x(x^2+1)}\,dx = \int \left(\frac{3}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx$

$$\frac{5x^2+x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$= 3\ln|x| + \ln(x^2+1) + \arctan x + C$

$5x^2+x+3 = A(x^2+1) + (Bx+C)x$

Let $x=0 \Rightarrow A=3$

Equate coeff

x^2: $5 = A+B \Rightarrow B=2$

x: $1 = C$

(b) [10] $\int \frac{2x+5}{x^2+4x+5}\,dx = \int \left(\frac{2x+4}{x^2+4x+5} + \frac{1}{(x+2)^2+1} \right) dx$

$= \ln|x^2+4x+5| + \arctan(x+2) + C$

3. [10] Evaluate. $\int_0^1 \ln x \, dx$ $= x \ln x \Big|_0^1 - \int_0^1 dx = (x \ln x - x)\Big|_0^1$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$= \lim_{R \to 0^+} \left[x \ln x - x \Big|_R^1 \right] = \lim_{R \to 0^+} \left[-1 - R \ln R + R \right]$

$= -1 - \lim_{R \to 0^+} R \ln R \quad \leftarrow \quad 0(-\infty)$ form

$= -1 - \lim_{R \to 0^+} \frac{\ln R}{1/R} \quad \leftarrow \quad \frac{-\infty}{\infty}$ form

$\overset{L'H}{=} -1 - \lim_{R \to 0^+} \frac{1/R}{-1/R^2} = -1 - \lim_{R \to 0^+} R$

$= -1$

4. [10] Find a constant C such that $p(x)$ is a probability density function on the given interval, and compute the probability indicated.

$$p(x) = \frac{C}{(x+1)^3} \quad \text{on}[0, \infty); \qquad P(0 \le X \le 1)$$

$1 = \int_0^\infty \frac{C}{(x+1)^3} dx = \lim_{R \to \infty} \frac{-C}{2} \frac{1}{(x+1)^2} \Big|_0^R = \lim_{R \to \infty} \left(\frac{-C}{2(R+1)^2} + \frac{C}{2} \right) = \frac{C}{2}$

so $C = 2$

$P(0 \le x \le 1) = \int_0^1 \frac{2}{(x+1)^3} dx = \frac{-1}{(x+1)^2} \Big|_0^1 = -\frac{1}{4} + 1 = 3/4$

5. [10] Use the Comparison Test to show $\int_1^\infty \frac{dx}{(x+2)(x+3)}$ converges.

Since $0 < \frac{1}{(x+2)(x+3)} < \frac{1}{x^2}$ & $\int_1^\infty \frac{dx}{x^2}$ is a convergent p-integral $(p=2>1)$,

by comparison $\int_1^\infty \frac{dx}{(x+2)(x+3)}$ also converges.

6. [10] Show that for $R > 1$,

$$\int_1^R \frac{dx}{(x+2)(x+3)} = \ln\left|\frac{R+2}{R+3}\right| - \ln\frac{3}{4}$$

Initially we note $\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$ so $1 = A(x+3) + B(x+2)$ $\quad x=-2 \Rightarrow A=1$, $x=-3 \Rightarrow B=-1$

so $\int_1^R \frac{dx}{(x+2)(x+3)} = \int_1^R \left[\frac{1}{x+2} - \frac{1}{x+3}\right] dx = \ln|x+2| - \ln|x+3| \Big|_1^R = \ln\left|\frac{x+2}{x+3}\right| \Big|_1^R$

$= \ln\left|\frac{R+2}{R+3}\right| - \ln\left(\frac{3}{4}\right)$

7. [5] Show $\int_1^\infty \frac{dx}{(x+2)(x+3)} = \ln\frac{4}{3}$

By above $\int_1^\infty \frac{dx}{(x+2)(x+3)} = \lim_{R\to\infty}\left(\ln\left|\frac{R+2}{R+3}\right| - \ln\left(\frac{3}{4}\right)\right) = \lim_{R\to\infty} \ln\left|\frac{1+2/R}{1+3/R}\right| + \ln\frac{4}{3}$

$= \ln 4/3$

8. [10] Find the arc length of $y = \dfrac{x^3}{12} + \dfrac{1}{x}$ for $1 \leq x \leq 2$. *Hint:* Show that $1 + (y')^2$ is a perfect square.

$$y' = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + (y')^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2$$

$$= 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}$$

$$= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$S = \int_1^2 ds = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= \frac{x^3}{12} - \frac{1}{x}\Bigg|_1^2$$

$$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{8 - 6 - 1 + 12}{12} = \frac{13}{12}$$

9. [10] Show the surface area of a sphere of radius R is $S = 4\pi R^2$ by computing the surface area generated by rotating the semicircle $y = \sqrt{R^2 - x^2}$ about the x-axis.

$$y' = \frac{1}{2}(R^2 - x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{R^2 - x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{R^2 - x^2}$$

$$= \frac{R^2 - x^2}{R^2 - x^2} + \frac{x^2}{R^2 - x^2}$$

$$= \frac{R^2}{R^2 - x^2}$$

so

$$ds = \frac{R}{\sqrt{R^2 - x^2}} dx$$

Surface Area $= \displaystyle\int_{-R}^{R} 2\pi \sqrt{R^2 - x^2}\, ds$

$$= \int_{-R}^{R} 2\pi R\, dx = 2\pi R x \Bigg|_{-R}^{R}$$

$$= 4\pi R^2$$