1. (Exercise 9.70, 1pt) MOM estimates the first population moment $EY$ with the first sample moment $\bar{Y}$. Since $EY = \lambda$ for a POI(\(\lambda\)) rv, the MOM estimator of $\lambda$ is $\hat{\lambda} = \bar{Y}$.

2. (Exercise 9.74, 2pts)
   a. First, calculate $EY = \int_0^\theta 2y(\theta - y) / \theta^2 dy = \theta/3$. Because MOM estimates the first population moment $EY = \theta/3$ with the first sample moment $\bar{Y}$, then the MOM estimator of $\theta$ is $\hat{\theta} = 3\bar{Y}$.
   b. The likelihood is $L(\theta) = 2^n \theta^{-2n} \prod_{i=1}^{n} (\theta - y_i)$. Therefore the likelihood can’t be factored into a function that only depends on $\bar{Y}$ and $\theta$ because, e.g., $\prod_{i=1}^{n} y_i / \theta^2$ is a term in the likelihood. Therefore, the MOM estimator is not a sufficient statistic for $\theta$.

3. (Exercise 9.80, 3pts)
   a. In class we found that the MLE for $\lambda$ was $\hat{\lambda} = \bar{Y}$.
   b. Because $E\bar{Y} = EY = \lambda$ then $E(\hat{\lambda}) = \lambda$. Because $V\bar{Y} = VY / n$ and $VY = \lambda$ then $V(\hat{\lambda}) = \lambda/n$.
   c. Since $\hat{\lambda}$ is unbiased and has a variance $V(\hat{\lambda}) = \lambda/n$ that goes to 0 as $n$ goes to infinity, then $\hat{\lambda}$ is consistent for $\lambda$.
   d. By the invariance property of MLEs, the MLE for $P(Y = 0) = \exp(-\lambda)$ is $\exp(-\hat{\lambda})$.

4. (2 pts) For iid $\text{EXP}(\theta)$ rvs, the likelihood is $L(\theta) = f(y_1, ..., y_n | \theta) = \frac{1}{\theta^n} e^{\sum y_i / \theta}$. The log likelihood is $\ln L(\theta) = -n \ln \theta - \sum y_i / \theta$ so $\frac{d}{d\theta} \ln L(\theta) = -\frac{n}{\theta} + \sum y_i / \theta^2$. Setting the derivative to zero shows that the MLE $\hat{\theta}$ satisfies $-n + \sum \frac{y_i}{\bar{y}} = 0$, so $\hat{\theta} = \bar{Y}$ is a critical point. The second derivative of the log likelihood is $\frac{d^2}{d\theta^2} \ln L(\theta) = \frac{n}{\theta^2} - 2 \sum y_i / \theta^3$. Evaluating at the critical point, $\frac{d^2}{d\theta^2} \ln L(\theta = \bar{Y}) = \frac{n}{\bar{y}^2} - 2 \sum \frac{y_i}{\bar{y}^2} = -\frac{n}{\bar{y}^2} < 0$. By the second derivative test $\hat{\theta} = \bar{Y}$ is the MLE for $\theta$.

5. (Exercise 9.81, 1pt) By #4, the MLE is $\hat{\theta} = \bar{Y}$. By the invariance property of MLEs, the MLE of $VY = 0^2$ is $\bar{Y}^2$.

6. (2 pts) See Example 9.15 on page 478 of your textbook, which shows that $\hat{\mu} = \bar{Y}$ is a critical point. Your book omits the second derivative test, which is easy: $\frac{d^2}{d\mu^2} \log(L(\mu)) = -n / \sigma^2$ is always negative. That is, $L(\mu)$ is a concave function. Thus, $\hat{\mu} = \bar{Y}$ is the MLE.
7. (Exercise 9.96, 2 pts) From Example 9.15, the MLE for \( \sigma^2 \) was found to be \( \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \).

(a) By the invariance property of MLEs, the MLE for \( \sigma = \sqrt{\sigma^2} \) is
\[
\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}.
\]

(b) Since \( \text{Var}(S^2) = 2\sigma^4/(n-1) \), then by the invariance property of MLEs, the MLE is \( 2 (S')^4/(n-1) \).

8. (Exercise 9.97, 1 pt)

a. MOM estimates the first population moment \( EY \) with the first sample moment \( \bar{Y} \). Since \( EY = 1/ \rho \), then the MOM estimator for \( \rho \) is \( \hat{\rho} = 1/\bar{Y} \).

b. We did this in class. The likelihood function is \( L(p) = p^n (1-p)^{\sum_{i=1}^{n} y_i - n} \) and the log–likelihood is
\[
\ln L(p) = n \ln p + (\sum_{i=1}^{n} y_i - n) \ln(1-p).
\]
Differentiating, we have
\[
\frac{d}{dp} \ln L(p) = \frac{n}{p} - \frac{1}{1-p} \left( \sum_{i=1}^{n} y_i - n \right). 
\]
Equating this to 0 and solving for \( p \), we obtain the MLE \( \hat{p} = 1/\bar{Y} \), which is the same as the MOM estimator found in part a. The 2nd derivative is
\[
\frac{d^2}{dp^2} \ln L(p) < 0 
\]
which shows that \( \hat{p} = 1/\bar{Y} \), is a maximizer.

9. (1 pt)
If \( Y \sim \text{Geometric}(p) \), then (from the back of the book), \( \text{Var}(Y) = t(p) = (1-p)/p^2 \). Problem #8b (Exercise 9.97b) showed that the MLE for \( p \) is \( \hat{p} = 1/\bar{Y} \). Thus, by the invariance property of MLEs, the MLE of \( \text{Var}(Y) = t(p) = (1-p)/p^2 \) is \( \hat{\text{Var}}(\bar{Y}) = \bar{Y}^2 (1-1/\bar{Y}) = \bar{Y}^2 - \bar{Y} \).

10. (Exercise 9.94, 1pt, required for grad students, EXTRA CREDIT otherwise)
Let \( \hat{\beta} = t(\hat{\theta}) \) so that \( \hat{\theta} = t^{-1}(\hat{\beta}) \). If the likelihood is maximized at \( \hat{\theta} \), then \( L(\hat{\theta}) \geq L(\hat{\theta}) \) for all \( \theta \). Define \( \hat{\beta} = t(\hat{\theta}) \) and denote the likelihood as a function of \( \beta \) as \( L(\beta) = L(t^{-1}(\beta)) \). Then, for any \( \beta \),
\[
L(\beta) = L(t^{-1}(\beta)) = L(\hat{\theta}) = L(t^{-1}(\hat{\beta})) = L(\hat{\beta}) .
\]
So, the MLE of \( \beta \) is \( \hat{\beta} \) and so the MLE of \( t(\theta) \) is \( t(\hat{\theta}) \).