Math 172 , Fall 2012 Section: ______ Name: SOLUTIONS

Final : Tuesday, December 11: 10-11:50am.

Problem	1	2	3	4	5	6	7	8	9	Total
Value	10	10	10	10	15	10	15	10	10	100
Points										

Instructions : All work must be shown to receive full credit.

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1. [10pts] Use integration by parts to evaluate the following integral:

$$I = \int x^2 \ln(x)\, dx$$

$u = \ln x \qquad v = \frac{1}{3}x^3$

$du = \frac{1}{x}dx \qquad dv = x^2 dx$

$$I = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$$I = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

2. [10pts] Evaluate the following integral:

$$I = \int \sin^3 x \ \cos^3 x\, dx$$

$$I = \int \sin^3 x\,(1-\sin^2 x)\,\underbrace{\cos x\, dx}_{du}$$

$u = \sin x$

$$I = \int u^3(1-u^2)\,du$$

$$I = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

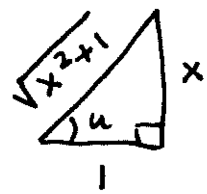
$$I = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

3. [10pts] Use a trigonometric substitution to evaluate:

$$I = \int \frac{1}{\sqrt{x^2+1}}\,dx$$

$$x = \tan u$$
$$dx = \sec^2 u \, du$$

$$I = \int \frac{\sec^2 u}{\sqrt{\tan^2 u + 1}}\, du$$



$$I = \int \sec u \, du$$

$$I = \ln|\sec u + \tan u|$$

$$I = \ln\left|\sqrt{x^2+1} + x\right| + C$$

4. [10pts] Use partial fraction expansions to evaluate:

$$I = \int \frac{2x}{(x+1)(x+3)}\,dx$$

$$\frac{2x}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+3)} = \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$$

Hence. $A+B=2$ and $3A+B=0 \Rightarrow A=-1, B=3$

$$I = \int \frac{3}{(x+3)} - \frac{1}{(x+1)}\,dx$$

$$I = 3\ln(x+3) - \ln(x+1) + C$$

5. [15pts] Conversion Problems: answer all three questions below

a) The curve given by $x = e^{2t}, y = 3e^{4t}$ lies on the graph $y = f(x)$. Find $f(x)$.

$$y = 3e^{4t} = 3(e^{2t})^2 = 3x^2$$

(Technically only $x > 0$)

$y = f(x) =$ $3x^2$

b) Express the polar equation $r = \frac{1}{cos\theta}$ in cartesian coordinates

$$r\cos\theta = 1$$

$$x = 1$$

$ANSWER:$ $x = 1$

c) A circle has cartesian equation $x^2 + y^2 = 2x$. Rexpress this in polar as $r = f(\theta)$.

$$x^2 + y^2 = 2x$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

$r = f(\theta) =$ $2\cos\theta$

6. [10pts] What is the slope $\frac{dy}{dx}$ of the line tangent to the curve

$$c(t) = (x(t), y(t)) = \left(sin(3t), \sqrt{t+4}\right)$$

at $t = 0$.

$$\left.\frac{dx}{dt}\right|_{t=0} = 3\cos(3t)\Big|_{t=0} = 3$$

$$\left.\frac{dy}{dt}\right|_{t=0} = \left.\frac{1}{2\sqrt{t+4}}\right|_{t=0} = \frac{1}{4}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{(\frac{1}{4})}{(3)} = \frac{1}{12}$$

$\frac{dy}{dx} =$ $\frac{1}{12}$

7. [15pts] Find the length of the curve defined by:

$$c(t) = \left(3t, 4t^{3/2}\right) \quad , \quad 0 \le t \le 2$$

$$c'(t) = (3, 6t^{1/2})$$

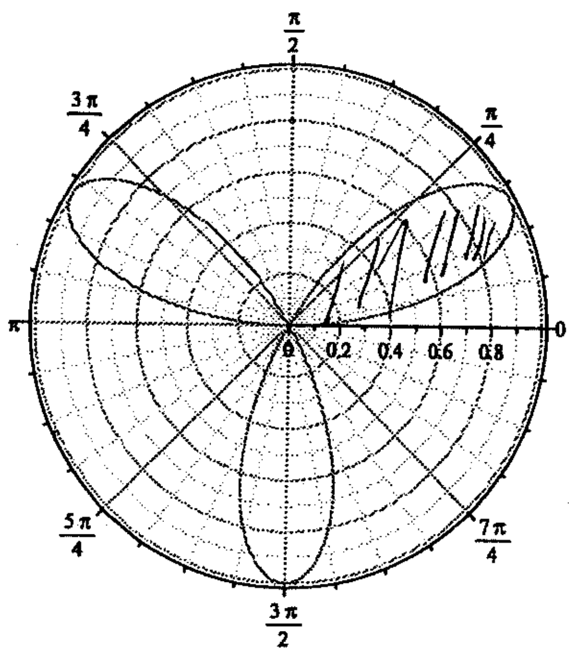
$$\|c'(t)\| = \sqrt{9 + 36t}$$

$$L = \int_0^2 \sqrt{9+36t}\, dt = \frac{1}{2}(1+4t)^{3/2}\Big|_0^2$$

$$L = \frac{1}{2}\left[9^{3/2} - 1\right] \qquad (9^{3/2} = 27)$$

$$L = 13$$

8. [10pts] Find the total area enclosed by the three-petal rose $r = \sin 3\theta$. Note the area is three times the area of the first petal.



$\sin 3\theta = 0$

$3\theta = 0, \pi, 2\pi, \ldots$

$\theta = 0, \pi/3, \frac{2\pi}{3}$

First petal has

$0 < \theta < \pi/3$

$$A = \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$A = \frac{3}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$$

$$A = \frac{3}{4}\left(\theta - \frac{1}{6}\cancel{\sin 6\theta}\right)\Big|_0^{\pi/3}$$

$$A = \frac{\pi}{4}$$

9. [10pts] A closed curve called a *cardioid* has polar equation $r = 1 - \cos\theta$ with $0 \le \theta \le 2\pi$.

a) Write out the definite integral whose value is the length of the cardioid.

$$A = 2\int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

$$A = 2\int_0^{\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$$A = 2\int_0^{\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

b) Evaluate the integral in part a). Hint: use common trigonometric identities including the half angle formula $\sin^2\left(\frac{z}{2}\right) = \frac{1}{2}(1 - \cos z)$ to simplify the integrand.

$$A = 2\sqrt{2}\int_0^{\pi} \sqrt{\underbrace{1 - \cos\theta}_{2\sin^2(\theta/2)}}\, d\theta$$

$$A = 4\int_0^{\pi} \sin(\theta/2)\, d\theta$$

$$A = -8\cos(\theta/2)\Big|_0^{\pi}$$

$$A = 8$$