Repeated Roots

Suppose we wish to solve

$$\vec{x}'(t) = A\vec{x}(t) \qquad A \in \mathbb{R}^{n\times n}$$

where A has a repeated real eigenvalue. Then the characteristic polynomial would look like

$$P(\lambda) = (\lambda - \lambda_1)^m (\lambda - \lambda_2) \cdots$$

Here λ_1 has "multiplicity" m.

EXAMPLE (Trivial to illustrate issue)

$$A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P(\lambda) = (\lambda - 1)^2$$

has $\lambda = 1$ eigenvalue with multiplicity $m = 2$

$$(A - \lambda I)\vec{\xi} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{\xi} \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has two independent eigenvectors. Get two solutions

$$\vec{x}_1 = e^{\lambda_1 t}\vec{\xi}_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x}_2 = e^{\lambda_1 t}\vec{\xi}_2 = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

EXAMPLE More typical

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \qquad P(\lambda) = (\lambda - 2)^2$$

$$(A - \lambda I)\vec{\xi} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \vec{0} \qquad \vec{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence we only have one eigenvector

$$\lambda = 2 \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{x}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Here $\lambda = 2$ has algebraic multiplicity 2 but geometric multiplicity 1 (one evector)

How do we find a 2nd independent soln?

Algebraic Multiplicity Two

$$P(\lambda) = (\lambda - \lambda_1)^2 (\lambda - \lambda_2)(\lambda - \lambda_3) \cdots$$

Suppose that

$$(A - \lambda_1 I)\vec{\xi} = \vec{0}$$

has a one dimensional solution space with eigenvector $\vec{\xi}_1$.

$$\vec{x}_1(t) = e^{\lambda_1 t}\vec{\xi}_1$$

is one solution of $\vec{x}' = A\vec{x}$ then.

We seek a soln of the form

(1) $$\vec{x}(t) = e^{\lambda t}\left(\vec{\eta}_0 t + \vec{\eta}_1\right)$$

Here $\vec{\eta}_0$ and $\vec{\eta}_1$ are to be determined

After some calculations one can show

$$A\vec{x} - \vec{x}' = e^{\lambda t}\left\{ \underbrace{(A-\lambda I)\vec{\eta}_0}\, t + \left[\underbrace{(A-\lambda I)\vec{\eta}_1 - \vec{\eta}_0}\right]\right\}$$

Must choose $\vec{\eta}_K$ so braced terms vanish.

(2) $$(A - \lambda I)\vec{\eta}_0 = \vec{0}$$

(3) $$(A - \lambda I)\vec{\eta}_1 = \vec{\eta}_0$$

So the procedure for finding a second soln is to find the eigenvector $\vec{\eta}_0$ then find any $\vec{\eta}_1$ which solves (3). Once found $\vec{x}_2(t)$ is found from (1)

<u>Remark</u> $\vec{\eta}_1$ called a generalized e-vector. It is a special solution of

$$(A - \lambda I)^2 \vec{\eta}_1 = \vec{0}$$

For higher dimensions one may need forms

$$\vec{x}(t) = e^{\lambda t}\left(\tfrac{1}{2}t^2\vec{\eta}_0 + t\vec{\eta}_1 + \vec{\eta}_2\right)$$

<u>EXAMPLE</u> Find two independent solns of $\vec{x}' = A\vec{x}$ if

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Calculations show $P(\lambda) = (\lambda - 2)^2 \Rightarrow \lambda = 2$ repeated

$$(A - \lambda I)\vec{\eta}_0 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\vec{\eta}_0 \qquad \vec{\eta}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so that one soln is

$$\vec{x}_1(t) = e^{2t}\vec{\eta}_0 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Next find a soln $\vec{\eta}_1$ of

$$(A - \lambda I)\vec{\eta}_1 = \vec{\eta}_0$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{\eta}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Such $\vec{\eta}_1$ are <u>not unique</u> since $\vec{\eta}_1 + \alpha\vec{\eta}_0$ soln too.

$$\vec{x}_2(t) = e^{2t}\left(\vec{\eta}_0 t + \vec{\eta}_1\right) = e^{2t}\begin{pmatrix} t \\ 1 - t \end{pmatrix}$$

Fundamental Matrix

$$X(t) = [\vec{x}_1\ \vec{x}_2] = \begin{bmatrix} e^{2t} & t e^{2t} \\ -e^{2t} & (1-t)e^{2t} \end{bmatrix}$$

EXAMPLE Two solutions of $\vec{x}' = A\vec{x}$ for

$$A = \begin{bmatrix} 8 & -4 \\ 1 & 4 \end{bmatrix}$$

Compute $P(\lambda) = \det(A - \lambda I) = (\lambda - 6)^2$. Hence $\lambda_1 = 6$ is a repeated e-value.

$$(A - \lambda_1 I)\vec{\eta}_0 = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\vec{\eta}_0 = \vec{0} \qquad \vec{\eta}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus one solution is

$$\vec{x}_1(t) = e^{\lambda_1 t}\vec{\eta}_0 = e^{6t}\vec{\eta}_0 = \begin{pmatrix} 2e^{6t} \\ e^{6t} \end{pmatrix}$$

Next find a solution $\vec{\eta}_1$ of

$$(A - \lambda_1 I)\vec{\eta}_1 = \vec{\eta}_0$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{\eta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Yields a second solution

$$\vec{x}_2(t) = e^{6t}\left(\vec{\eta}_0 t + \vec{\eta}_1\right) = e^{6t}\begin{pmatrix} 2t + 1 \\ t \end{pmatrix}$$

Fundamental matrix $\mathbf{X}(t) = [\vec{x}_1 \vdots \vec{x}_2]$

$$\mathbf{X}(t) = \begin{bmatrix} 2e^{6t} & e^{6t}(2t+1) \\ e^{6t} & te^{6t} \end{bmatrix}$$

Variation of Parameters

$$(1) \qquad \vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$$

Given we have a Fundamental matrix $\mathbf{X}(t)$ for the homogeneous ($f=0$) case we seek a solution of (1) of the form

$$(2) \qquad \vec{x}_p = \mathbf{X}(t)\vec{v}(t)$$

Here $\vec{x}_p(t)$ is a particular solution. Once $\vec{v}(t)$ is found the general solution is

$$\vec{x}(t) = \mathbf{X}(t)\vec{c} + \vec{x}_p(t)$$

To find $\vec{v}(t)$ we make use of

$$\mathbf{X}' = A\mathbf{X}$$

Sub (2) into (1)

$$\mathbf{X}'\vec{v} + \mathbf{X}\vec{v}' = A\mathbf{X}\vec{v} + \vec{f}$$

$$\cancel{A\mathbf{X}\vec{v}} + \mathbf{X}\vec{v}' = \cancel{A\mathbf{X}\vec{v}} + \vec{f}$$

$$\mathbf{X}\vec{v}' = \vec{f}$$

$$\vec{v}' = \mathbf{X}^{-1}\vec{f}$$

$$\vec{v} = \int \mathbf{X}(t)^{-1}\vec{f}(t)\,dt$$

Hence

$$(3) \qquad \vec{x}_p(t) = \mathbf{X}(t)\int^{t} \mathbf{X}(s)^{-1}\vec{f}(s)\,ds$$

EXAMPLE

$$A = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \qquad \vec{f}(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$$

Can show $P(\lambda) = \lambda^2 - 1$ so $\lambda = \pm 1$ and two independent solns are

$$\vec{x}_1(t) = e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{x}_2(t) = e^{-t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Thus the Fundamental Matrix is

$$\mathbf{X}(t) = \begin{bmatrix} 0 & 2e^{-t} \\ e^{t} & -3e^{-t} \end{bmatrix}$$

The Wronskian $W(t) = \det \mathbf{X}(t) = -2$ can be used to find the inverse

$$\mathbf{X}(t)^{-1} = -\frac{1}{2}\begin{bmatrix} -3e^{-t} & -2e^{-t} \\ -e^{t} & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{-t} & e^{-t} \\ \frac{1}{2}e^{t} & 0 \end{bmatrix}$$

Then

$$\mathbf{X}(t)^{-1}\vec{f}(t) = \begin{bmatrix} 2 \\ \frac{1}{2}e^{2t} \end{bmatrix}$$

$$\vec{v}(t) = \int^{t} \begin{pmatrix} 2 \\ \frac{1}{2}e^{2s} \end{pmatrix} ds = \begin{pmatrix} 2t \\ \frac{1}{4}e^{2t} \end{pmatrix}$$

and then $\vec{x}_p(t)$ from

$$\vec{x}_p(t) = \mathbf{X} v = \begin{bmatrix} 0 & 2e^{-t} \\ e^{t} & -3e^{-t} \end{bmatrix}\begin{bmatrix} 2t \\ \frac{1}{4}e^{2t} \end{bmatrix}$$

$$\vec{x}_p(t) = \begin{pmatrix} \frac{1}{2}e^{2t} \\ (2t - \frac{3}{4})e^{t} \end{pmatrix}$$