Laplace Transform Review

QUESTION ONE Find $\mathcal{L}\{f(t)\}$ if

a) $f(t) = t^2 e^{3t}$

b) $f(t) = e^{2t}\sin(3t)$

c) $f(t) = H(t-4)$

d) $f(t) = \int_0^t e^{\lambda}\sin(t-\lambda)\,d\lambda$

e) $f(t) = t(t-1)$

f) $f(t) = t\,y'(t)$ where $Y(s) = \mathcal{L}\{y(t)\} = \frac{1}{s^{3/2}}$

ANSWERS

a) $F(s) = \frac{2}{(s-3)^3}$ b) $F(s) = \frac{3}{(s-5)(s+1)}$

c) $F(s) = \frac{e^{-4s}}{s}$ d) $F(s) = \frac{1}{(s-1)(s^2+1)}$

e) $F(s) = \frac{2}{s^3} - \frac{1}{s^2}$ f) $F(s) = -\frac{1}{2}\frac{1}{s^{3/2}}$

The last follows from

$$F(s) = \frac{d}{ds}\left(sY(s) - y(0)\right)$$

QUESTION TWO

By completing the square and/or using partial fraction expansions, invert the following transforms $F(s)$ of $f(t)$

a) $$F(s) = \frac{1}{s^2+s-2}$$

b) $$F(s) = \frac{s+1}{s^2+s-2}$$

c) $$F(s) = \frac{s}{s^2+2s+5}$$

d) $$F(s) = \frac{s-3}{s^2+6s}$$

e) $$F(s) = \frac{1}{(s-1)(s^2+4)}$$

ANSWERS

a) $F = \frac{1}{3}(s-1)^{-1} - \frac{1}{3}(s+2)^{-1}$ $\quad f = \frac{1}{3}(e^{t} - e^{-2t})$

b) $F = \frac{2}{3}(s-1)^{-1} + \frac{1}{3}(s+2)$ $\quad f = \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t}$

c) $F = \dfrac{s}{(s+1)^2+4}$ $\quad f = \frac{1}{2}e^{-t}(2\cos 2t - \sin 2t)$

d) $F = -\dfrac{1}{2s} + \dfrac{3}{2(s+6)}$ $\quad f = -\frac{1}{2} + \frac{3}{2}e^{-6t}$

e) $F = \dfrac{1}{5(s-1)} - \dfrac{(s+1)}{5(s^2+4)}$ $\quad f = \frac{1}{5}e^{t} - \frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t)$

QUESTION THREE Solve the IVP using Laplace Transforms.

a) $y'' - y = 2\cos t \qquad y(0) = 0 \qquad y'(0) = 1$

b) $y'' - 2y' + y = e^t \qquad y(0) = 0 \qquad y'(0) = 1$

c) $y'' - 2y' + 2y = e^t \qquad y(0) = 1 \qquad y'(0) = 0$

d) $y'' - y' - 2y = 4t^2 \qquad y(0) = 0 \qquad y'(0) = -1$

e) $y'' - 2y' + 2y = h(t) \qquad y(0) = 0 \qquad y'(0) = 0$

ANSWERS

a) $y = e^t - \cos t$

b) $y = \frac{1}{2}e^t(2t + t^2) \qquad Y(s) = \frac{s}{(s-1)^3}$

c) $y = e^t(1 - \sin t) \qquad Y = \frac{1}{(s-1)} - \frac{1}{(s^2 - 2s + 2)}$

d) $y = -2t^2 + 2t - 3 + 3e^{-t}$ Long hand Partial Fraction

e) $$y(t) = \int_0^t h(\lambda)\, e^{(t-\lambda)} \sin(t-\lambda)\, d\lambda$$

QUESTION FOUR Compute the convolution of $f(t)$ with $g(t)$

a) $f(t) = e^{t}$, $g(t) = e^{-t}$

b) $f(t) = t^{2}$, $g(t) = t$

ANSWERS

a) $(f*g)(t) = \sinh(t)$

b) $(f*g)(t) = \frac{1}{12} t^{4}$

QUESTION FIVE Find the transform of $f(t)$ using the definition

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

a) $$f(t) = \begin{cases} e^{t} & t \leq 2 \\ 0 & t > 2 \end{cases}$$

ANSWER

$$F(s) = \frac{1 - e^{2(1-s)}}{s-1}$$

QUESTION SIX Use convolution theorem to invert

$$F(s) = \frac{1}{(s-2)(s-3)} = \frac{1}{(s-2)} \frac{1}{(s-3)}$$

$$f(t) = \int_0^t e^{2t-2\lambda} e^{3\lambda}\,d\lambda = e^{3t} - e^{2t}$$