Linear First Order

$\mu(x) = \exp\left(\int p(x)\,dx\right)$ int. factor

1) $y' + 3y = 1$ $\quad \mu = e^{3x}$ $\quad y = \frac{1}{3} + c_1 e^{-3x}$

2) $y' + \frac{3}{x} y = \sqrt{x}$ $\quad \mu = x^3$ $\quad y = \frac{2}{9} x^{3/2} + c_1 x^{-3}$

3) $y' - \frac{1}{2x} y = \sqrt{x}$ $\quad \mu = \frac{1}{\sqrt{x}}$ $\quad y = x^{3/2} + c_1 \sqrt{x}$

4) $y' + \tan x \, y = \sin x$ $\quad \mu = \sec x$ $\quad y = -\cos x \ln|\cos x| + c_1 \cos x$

5) $y' + \frac{2x}{1+x^2} y = \frac{1}{x^2+1}$ $\quad \mu = x^2+1$ $\quad y = \frac{x}{x^2+1} + \frac{c_1}{x^2+1}$

Bernoulli

$y' + P(x) y = y^n, \quad v = y^{1-n}$

1) $y' + y = y^3$ $\quad v = y^{-2}$ $\quad y = (1 + c_1 e^{-2x})^{-1/2}$

2) $y' + \frac{1}{x} y = y^2$ $\quad v = y^{-1}$ $\quad y = (c_1 x - x \ln x)^{-1}$

3) $y' - \frac{1}{x} y = y^2$ $\quad v = y^{-1}$ $\quad y = \frac{2x}{c_1 - x^2}$

4) $y' + \frac{1}{x} y = 5x y^{-2}$ $\quad v = y^3$ $\quad y = \left(\frac{3x^5 + c_1}{x^3}\right)^{1/3}$

5) $y' + \frac{2}{3(x+1)} y = 4(x+1) y^{-1/2}$ $\quad y = \left(\frac{2(x+1)^3 + c_1}{x+1}\right)^{2/3}$

EXACT EQUATIONS $\quad M\,dx + N\,dy = F_x\,dx + F_y\,dy = 0$

1) $$2xy^3\,dx + (3x^2y^2 + 2y)\,dy = 0$$

$$F(x,y) = x^2y^3 + y^2 = C \quad \text{imp. soln}$$

2) $$(\sec^2 x + 3)\,dx + (3 + \tan^2 y)\,dy = 0$$

$$F(x,y) = \tan x + 3x + 2y + \tan y = C \quad \text{imp. soln}$$

3) $$\left(1 + \frac{1}{2\sqrt{x+y^2}}\right)dx + \left(2 + \frac{y}{\sqrt{x+y^2}}\right)dy = 0$$

$$F(x,y) = x + 2y + \sqrt{x+y^2}$$

4) $$\left(x + y - \frac{1}{x}\right)dx + (x+y)\,dy = 0$$

$$F(x,y) = \tfrac{1}{2}(x+y)^2 - \ln x = c$$

after completing square. Soln is implicit.

If additionally $y(1) = 1$ then $F(1,1) = c = 2$

$$\tfrac{1}{2}(x+y)^2 - \ln x = 2$$

Solving for $y(x)$ yields an explicit soln

$$y(x) = -x + \sqrt{4 + 2\ln x}$$

Homogenous Equations

$$\frac{dy}{dx} = f(x, y) = G(v) \qquad v = \frac{y}{x}$$

Leads to

$$\int \frac{dv}{G(v) - v} = \int \frac{dx}{x}$$

1) $$f(x,y) = \frac{x^2 + y^2 + xy}{x^2} \qquad y = x \tan(\ln x + c_1)$$

2) $$f(x,y) = \frac{x + y \sin(y/x)}{x \sin(y/x)} \qquad G(v) = \frac{1 + v \sin v}{\sin v}$$

$$y = x \arccos(c_1 - \ln x)$$

3) $$f(x,y) = \frac{x^2 + y^2}{xy} \qquad G(v) = \frac{1}{v} + v$$

$$y = \pm x\sqrt{c_1 + 2\ln x}$$

4) $$f(x,y) = \frac{x\sqrt{x^2+y^2} + y^2}{xy} \qquad G(v) = v + \frac{1}{v}\sqrt{v^2+1}$$

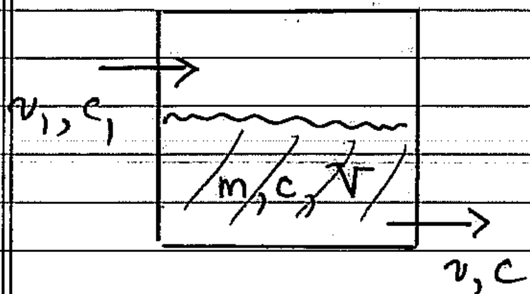
Solution in terms of $v(x)$

$$\sqrt{v^2+1} = \ln x + c_1$$

Hence

$$y(x) = \pm\sqrt{(\ln x + c_1)^2 - 1}$$

Mixing Problems



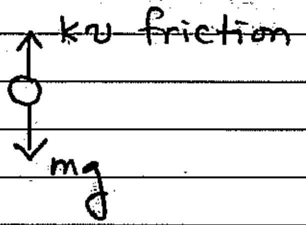
Volume $V(t) = V_0 + (v_1 - v)t$

$$\frac{dm}{dt} = v_1 c_1 - \frac{v m}{V(t)}$$

Ex If no mass initially, $V_0 = 100\,L$, $v_1 = 10\,L/min$, $c_1 = 0.1\,kg/L$, $v = 5\,L/min$ then

$$m(t) = \frac{t(40+t)}{2(20+t)} \quad \text{after simplifying}$$

Newtonian Mechanics



$$m\frac{dv}{dt} = mg - kv \qquad v(0) = v_0$$

Ex $v_0 = 0$, $m = 10\,kg$, $g = 9.8\,m/sec$, $k = 5$ then

$$\frac{dv}{dt} = g - \frac{1}{2}v \qquad v(t) = 2g\left(1 - e^{-t/2}\right)$$

Rocket Given mass $m(t)$ due to expulsion of fuel

$$\frac{d}{dt}(mv) = -mg \qquad m(t) = m_0 - \alpha t \quad (\text{mass } m, \text{ rate } \alpha)$$

easiest to integrate as

$$mv = -\int mg\,dt + C$$

Ex $m_0 = 1000\,kg$, $v(0) = 0$, $\alpha = 10\,kg/sec$

$$v(t) = \frac{(100t - \frac{1}{2}t^2)}{t - 100}\, g$$