Second Order Review Problems

1) Find the general soln of

$y'' - 3y' + 2y = 0$ $\qquad$ $y = c_1 e^x + c_2 e^{2x}$

$y'' + 2y' + y = 0$ $\qquad$ $y = c_1 e^{-x} + c_2 x e^{-x}$

$y'' - 2y' + 5y = 0$ $\qquad$ $y = c_1 e^x \sin(2x) + c_2 e^x \cos(2x)$

$y'' \quad + 7y = 0$ $\qquad$ $y = c_1 \cos\sqrt{7}x + c_2 \sin\sqrt{7}\, x$

2) Given $y = e^x$ is a soln of

$$y''' - y'' + y' - y = 0$$

find its general soln.

Since $r = 1$ is root $P(r) = r^3 - r^2 + r - 1$
long division yields $P = (r-1)(r^2+1)$ implies

$$y(x) = c_1 e^x + c_2 \sin x + c_3 \cos x$$

3) Given $y = e^{-x}$ is a soln of

$$y''' - 3y'' + 4y = 0$$

Find its general soln. Here $(r+1)$ a factor of charact. polynomial. Long division

$$P(r) = (r+1)(r-2)^2$$

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}$$

4) Find the general solns of the Cauchy Euler Eqn:

$x^2y'' - 2xy' + 2y = 0$ $\quad y = c_1x + c_2x^2$

$2x^2y'' + 3xy' - y = 0$ $\quad y = c_1x^{-1} + c_2\sqrt{x}$

$x^2y'' - 3xy' + 4y = 0$ $\quad y = c_1x^2 + c_2x^2\ln x$

$x^2y'' - 3xy' + 20y = 0$ $\quad y = c_1x^2\cos(4\ln x) + c_2x^2\sin(4\ln x)$

$x^2y'' + xy' + 3y = 0$ $\quad y = c_1\cos(\sqrt{3}\ln x) + c_2\sin(\sqrt{3}\ln x)$

5) Use undetermined coeff method to find a particular soln y_p. Roots of $P(r)$ indicated.

$y'' + 3y = 2$ $\quad y_p = A = \frac{2}{3}$ $\quad r = \pm\sqrt{3}i$

$y'' - 3y' + 2y = x + 1$ $\quad y_p = \frac{1}{2}x + \frac{5}{4}$ $\quad r = 1, 2$

$y'' - 3y' + 2y = e^x$ $\quad y_p = Axe^x = -xe^x$ $\quad r = 1, 2$

$y'' - 2y' + y = 5\sin(2x)$ $\quad y_p = \frac{4}{5}\cos(2x) - \frac{3}{5}\sin(2x)$

$y'' - 2y' + y = e^x$ $\quad y_p = Ax^2e^x$ $\quad$ since $r = 1$ repeated twice

$y_p = \frac{1}{2}x^2e^x$

6) Compute the <u>Wronskian</u> of the independent homogeneous solns

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = 0$ $\quad$ $y_1 = x$ $\quad$ $y_2 = x \ln x$

$W = x$

$y'' + 2y' + y = 0$ $\quad$ $y_1 = e^{-x}$ $\quad$ $y_2 = xe^{-x}$

$W = e^{-2x}$

$y'' + 9y = 0$ $\quad$ $y_1 = \cos(3x)$ $\quad$ $y_2 = \sin(3x)$

$W = 3$

7) <u>Variation of parameters</u>

Eqn must be in standard form

$$L(y) = y'' + p(x)y' + q(x)y = f(x)$$

Let y_1, y_2 be homogeneous solns with wronskian $W(x)$. Integrate

$$v_1' = -\frac{f(x)y_2(x)}{W(x)} \qquad v_2' = +\frac{f(x)y_1(x)}{W(x)}$$

then

$$y_p = v_1 y_1 + v_2 y_2$$

is a particular solution of $L(y) = f$.

a) Find a particular soln of (Var of Param)

$$y'' - y = e^x$$

Here $y_1 = e^x$, $y_2 = e^{-x}$, $W = -2$ and

$$v_1' = \frac{1}{2} \qquad v_2' = -\frac{1}{2}e^{2x}$$

yields $y_p = \frac{1}{4}e^x(2x-1)$

b) Find a particular soln of

$$y'' + \frac{1}{x}y' = \frac{\ln x}{x^2}$$

Here $y_1 = \ln x$, $y_2 = 1$, $W = -\frac{1}{x}$ so

$$v_1' = \frac{\ln x}{x} \qquad v_2' = -\frac{(\ln x)^2}{x}$$

yields $y_p = \frac{1}{6}(\ln x)^3$

c) Find a particular soln of

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{1}{x^3}$$

Here $y_1 = x\ln x$, $y_2 = x$, $W = -x$ so

$$v_1' = \frac{1}{x^3} \qquad v_2' = -\frac{\ln x}{x^3} \quad \text{(Int. By Part)}$$

yields

$$y_p = \frac{1}{4x}$$

after simplifying. Kinda nasty one.

d) Particular Soln of

$$y'' - 2y' + y = x^3 e^x$$

Homog solns $y_1 = e^x$, $y_2 = xe^x$, $W = e^{2x}$.

$$v_1' = -x^4 \qquad v_2' = x^3$$

yields $y_P = \frac{1}{20} x^5 e^x$

e) Particular Soln of

$$y'' + y = \sec x \tan x$$

Here $y_1 = \cos x$, $y_2 = \sin x$, $W = +1$

$$v_1' = -\frac{\sin^2 x}{\cos^2 x} \qquad v_2' = \frac{\sin x}{\cos x}$$

First integral use $\sin^2 x = 1 - \cos^2 x$ and $(\tan x)' = \sec^2 x$

$$y_P = (x - \tan x)\cos x - \sin x \ln(\cos x)$$

8) Initial Value Problems (IVP)

In each of the following you are only given the particular soln $y_p(x)$. Find the unique soln to the IVP.

a) $y'' + 2y' + y = 1$ $\qquad y_p = 1$

$y(0) = 1 \qquad y'(0) = 2$

b) $y'' - y = x$ $\qquad y_p = -x$

$y(0) = y'(0) = 0$

c) $x^2 y'' + x y' + y = f(x)$ $\qquad y_p = \sin(\pi x)$

$y(1) = 2 \qquad y'(1) = 0$

ANSWERS

a) $y(x) = 1 + 2xe^{-x}$

b) $y(x) = \frac{1}{2}(e^{x} - e^{-x}) - x \quad = \sinh x - x$

c) $y(x) = \pi \sin(\ln x) + 2\cos(\ln x) + \sin(\pi x)$

Note that in c) you don't need to know $f(x)$ but you can compute it since $f(x) = L(y_p)$ and y_p is given.

9) Reduction of Order (Standard Form)

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

Suppose you know $y_1(x)$ is a soln of the homogeneous problem

$$y_2(x) = y_1(x)\, v(x)$$

is a second solution by the theory if

$$v(x) = \int^x \frac{\exp\left(-\int p(t)\,dt\right)}{y_1(t)^2}\, dt$$

EX Find y_2 if $y_1(x) = x$ is the known soln of

$$L(y) = y'' \underbrace{- \frac{(x+2)}{x}}_{p} y' + \frac{(x+2)}{x^2} y = 0$$

Can show

$$\mu = \exp\left(-\int p(t)\,dt\right) = t^2 e^t$$

So that

$$v(x) = \int^x \frac{t^2 e^t}{(t^2)}\, dt = e^x$$

and

$$y_2(x) = x\, v(x) = x e^x$$

Ex Let $y_1(x) = \sqrt{x+1}$ be a soln of

$$y'' + q(x)\,y = 0$$

for some $q(x)$. Find second independent soln. Here $p(x) \equiv 0$.

$$\mu = \exp\left(-\int p(t)\,dt\right) = \exp(0) = 1$$

Hence $y_2 = y_1 v$ where

$$v(x) = \int^x \frac{1}{y_1(t)^2}\,dt = \int^x \frac{1}{t+1}\,dt = \ln(x+1)$$

and $y_2(x) = \sqrt{x+1}\,\ln(x+1)$

Ex Let $y_1(x) = \sqrt{\tan x}$ be a soln of

$$y'' + q(x)\,y = 0$$

Find y_2 (don't need to know q). Again $\mu = 1$.

$$v(x) = \int^x \frac{1}{\tan t}\,dt = \ln(\sin x)$$

so $y_2(x) = \sqrt{\tan x}\,\ln(\sin x)$

Ex Let $y_1(x) = \sqrt{x+1}$ solve (for some $q(x)$)

$$y'' - \frac{1}{x}y' + q(x)\,y = 0$$

Show

$$y_2(x) = \sqrt{x+1}\,\left(x - \ln(x+1)\right)$$

10) <u>Amplitude Phase Form</u>

$$my'' + by' + ky = 0$$

Oscillations only if $b^2 - 4mk < 0$

$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

(1) $$y = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

where roots of characteristic polynomial

$$r = \alpha \pm i\beta \qquad \alpha = -\frac{b}{2m} \qquad \beta = \frac{1}{2m}\sqrt{4mk - b^2}$$

Here β is frequency, $T = 2\pi/\beta$ is period.
Eqn (1) can be written

$$y = e^{\alpha t} \cdot A\sin(\beta t + \phi)$$

where

(2) $$A\sin\phi = c_1 \qquad A\cos\phi = c_2$$

Find c_1, c_2 from Init. Cond then use (2) to compute A, ϕ

$$A = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \arctan\left(c_1/c_2\right)$$

EX Mass $m = 2\text{kg}$ with friction coeff $b = 2$ (units N-sec/m) and $k = 1$ N/m.
Initially $y(0) = 1$, $y'(0) = 0$

$$2y'' + 2y' + y = 0$$

Has general soln

$$y(t) = e^{-t}\left(c_1 \cos(\tfrac{1}{2}t) + c_2 \sin(\tfrac{1}{2}t)\right)$$

Initial conditions yield $c_1 = c_2 = 1$.

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{2}$$

$$\tan\phi = \frac{c_1}{c_2} = 1 \quad \Rightarrow \quad \phi = \pi/4$$

and solution is

$$y(t) = \sqrt{2}\, e^{-t} \sin\left(\tfrac{1}{2}t + \tfrac{\pi}{4}\right)$$

has period $T = \frac{2\pi}{\beta} = 4\pi$.