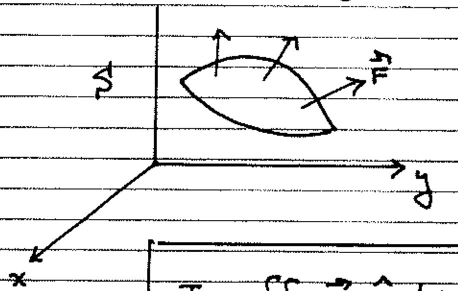
FLUX Surface Integrals



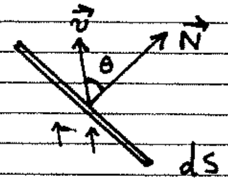
$\vec{F} = \langle F_1, F_2, F_3 \rangle$

vector field of "stuff"

$$\boxed{\Phi = \iint_S \vec{F}\cdot\hat{N}\, dS = \text{flux of } \vec{F} \text{ thru } S}$$

Here $\hat{N}$ is a unit normal vector.
Since there are two such vectors
one must specify which as in up/down
or inward/outward.

To get an idea of what flux Φ means
suppose a fluid of density ρ and
vellocity $\vec{v} = \langle v_1, v_2, v_3 \rangle$ flows thru
a surface S. Consider the mass flow
rate thru a surface element dS



$\vec{N} = \langle -f_x, -f_y, 1 \rangle$

Were $\theta = \frac{\pi}{2}$ no fluid would flow thru dS
Were $\theta = 0$ maximal fluid flow rate thru dS

What matters is the normal component of $\vec{v}$

The normal component is $\vec{v}\cdot\hat{N}$ where
$\hat{N}$ is the unit normal.

In time Δt the volume of the fluid
thru dS is

$$\Delta V = \underbrace{(\vec{v}\cdot\hat{N})}_{m/sec}\ \underbrace{dS}_{m^2}\ \underbrace{\Delta t}_{sec}$$

Thus the net rate of mass thru dS is

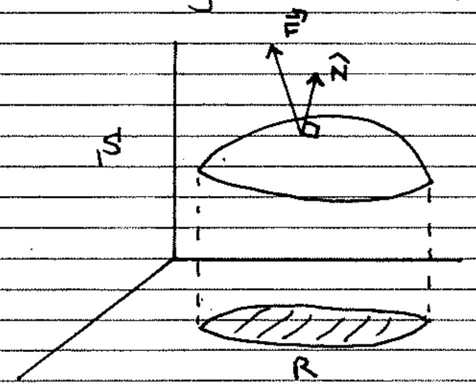
$$\frac{\Delta M}{\Delta t} = (\rho\vec{v})\cdot\hat{N}\, dS$$

Add up this on the entire surface to get

$$\frac{dM}{dt} = \Phi = \iint_S \vec{F}\cdot\hat{N}\, dS$$

where $\vec{F} = \rho\vec{v}$. Generally $\vec{F}$ is any
field, electric, magnetic, gravitational
or otherwise.

Calculating fluxes (Graphs)



Surface defined by

$z = f(x,y)$

where $(x,y) \in R$

$\vec{F} = \langle F_1, F_2, F_3 \rangle$ force field

$\vec{N} = \langle -f_x, -f_y, 1 \rangle$ normal up

$dS = \|\vec{N}\|\, dA$ area element

Then

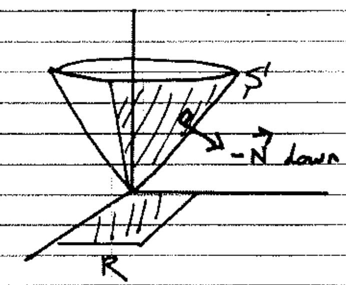
$$\Phi = \iint_S \vec{F}\cdot\hat{N}\, \underline{\underline{dS}} = \iint_R (\vec{F}\cdot\hat{N})\, \underline{\underline{\|\vec{N}\|\, dA}}$$

Summary

$$\boxed{\Phi = \iint_S \vec{F}\cdot\hat{N}\, dS = \iint_R (\vec{F}\cdot\vec{N})\Big|_{z=f(x,y)} dA}$$

Ex Flux of $\vec{F} = \langle x, y, z \rangle$ thru portion of the cone $z = \sqrt{x^2+y^2}$ over $0 \le x, y \le 1$

up $\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle -\frac{x}{r}, -\frac{y}{r}, 1 \rangle$



$\vec{F}\cdot\vec{N} = \langle x, y, z \rangle \cdot \langle -\frac{x}{r}, -\frac{y}{r}, 1 \rangle$

$\vec{F}\cdot\vec{N} = -\frac{(x^2+y^2)}{r} + z$

$\vec{F}\cdot\vec{N} = -r + z$

But on the surface
$z = \sqrt{x^2+y^2} = r$.

$\vec{F}\cdot\vec{N} \quad = -r + z$

$\vec{F}\cdot\vec{N}\Big|_{z=r} = -r + r \quad = 0$

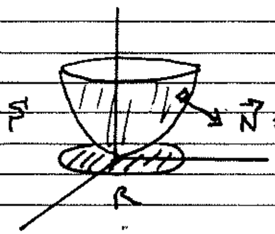
Hence flux

$$\Phi = \int_0^1 \int_0^1 0 \, dy\, dx$$

$$\Phi = 0$$

If you look closely you'll discover $\vec{F}$ is always tangent to S hence <u>no flux</u>

<u>Ex</u> Flux of $\vec{F} = \langle x, y, z\rangle$ thru portion of paraboloid $z = x^2+y^2$ inside cylinder $x^2+y^2 \le 1$. Orient downward.



$\vec{N} = \langle +f_x, +f_y, -1\rangle$ downward $\vec{N}$

Since $z = f(x,y) = x^2 + y^2$

$$\vec{N} = \langle f_x, f_y, -1\rangle = \langle 2x, 2y, -1\rangle$$

$$\vec{F}\cdot\vec{N} = \langle x, y, z\rangle \cdot \langle 2x, 2y, -1\rangle$$

$$\vec{F}\cdot\vec{N} = 2x^2+2y^2 - z$$

$$\vec{F}\cdot\vec{N} = 2r^2 - z$$

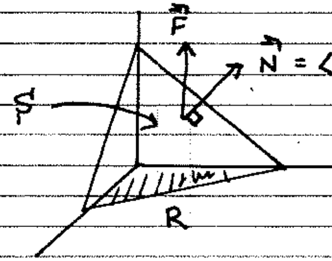
But $z = r^2$ on S

$$\vec{F}\cdot\vec{N}\Big|_{z=r^2} = 2r^2 - r^2 = r^2$$

and

$$\Phi = \int_0^{2\pi}\int_0^1 \underbrace{r^2}_{\vec{F}\cdot\vec{N}|_{z=f}} \underbrace{r\,dr\,d\theta}_{dA} = \cdots = \frac{\pi}{2}$$

<u>Ex</u> Flux of $\vec{F} = \langle z, 1, z\rangle$ through portion of $z = 1-x-2y$ in first octant $(x,y,z \ge 0)$. Orient upward.



$\vec{N} = \langle -f_x, -f_y, 1\rangle$

Plane intersects $z=0$

$$0 = 1-x-2y$$

Hence $y = \frac{1}{2}(1-x)$, $x\in(0,1)$ give limits of integration.

$$\vec{N} = \langle 1, 2, 1\rangle$$

$$\vec{F}\cdot\vec{N} = \langle z, 1, z\rangle\cdot\langle 1, 2, 1\rangle$$

$$\vec{F}\cdot\vec{N} = 2z+2$$

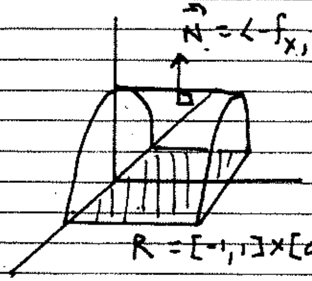
$$\vec{F}\cdot\vec{N}\Big|_S = 2 + 2(1-x-2y) = 4-2x-4y$$

Integrate over triangle R

$$\Phi = \int_0^1\int_0^{\frac{1}{2}(1-x)} (4-2x-4y)\,\underbrace{dy\,dx}_{dA} = \cdots = \frac{2}{3}$$

Ex Flux of $\vec{F} = \langle x^2 z, y, z \rangle$ thru portion of $z = 1 - y^2$ over $[-1,1] \times [0,1]$ oriented upward.

$\vec{N} = \langle -f_x, -f_y, 1 \rangle$



$R = [-1,1] \times [0,1]$

Here $z = 1 - y^2$

$\vec{N} = \langle -f_x, -f_y, 1 \rangle$

$\vec{N} = \langle 0, 2y, 1 \rangle$

$$\vec{F} \cdot \vec{N} = \langle x^2 z, y, z \rangle \cdot \langle 0, 2y, 1 \rangle$$

$$\vec{F} \cdot \vec{N} = 2y^2 + z$$

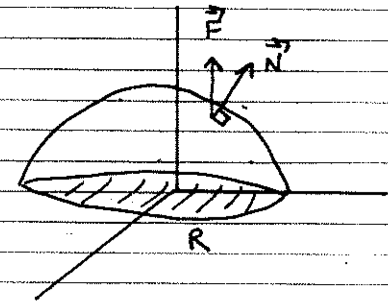
Then on the surface $z = 1 - y^2$

$$\vec{F} \cdot \vec{N} = y^2 + 1$$

So that

$$\Phi = \int_{-1}^{1} \int_{0}^{1} (1 + y^2)\, dy\, dx = \frac{8}{3}$$

Ex Flux of $\vec{F} = 10\hat{k}$ thru the hemisphere $z = \sqrt{1 - x^2 - y^2}$ oriented upward



$\vec{F} = \langle 0, 0, 10 \rangle$

$\vec{N} = \langle -f_x, -f_y, 1 \rangle$

You could calculate $\vec{N}$ but what matters is $\vec{F} \cdot \vec{N}$ only!

$$\vec{F} = \langle 0, 0, 10 \rangle$$

$$\vec{N} = \langle -f_x, -f_y, 1 \rangle$$

$$\vec{F} \cdot \vec{N} = 10$$

Hence flux is

$$\Phi = \iint_R 10\, dA = 10 \iint_R dA$$

$\Phi = 10$ (area of circle radius one)

$$\Phi = 10\pi$$

Ex Flux of $\vec{F} = \langle -y, x, z \rangle$ thru portion of $z = xy$ inside $x^2+y^2 \le 4$ and $x, y > 0$. Oriented upward.

$$\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle -y, -x, 1 \rangle \qquad \text{up.}$$

$$\vec{F} \cdot \vec{N} = \langle -y, -x, z \rangle \cdot \langle -y, -x, 1 \rangle$$

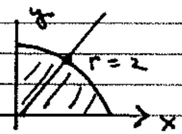
$$\vec{F} \cdot \vec{N} = x^2 + y^2 + z$$

$$\vec{F} \cdot \vec{N} = r^2 + z$$

on surface

$$\vec{F} \cdot \vec{N} \Big|_S = r^2 + xy = r^2 + r^2 \sin\theta \cos\theta$$

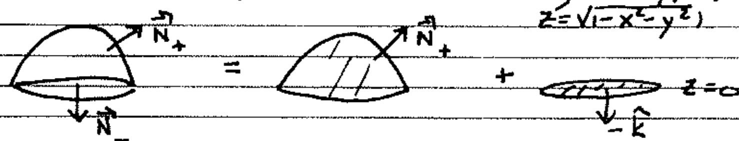
Region



$0 < r < 2$
$0 < \theta < \pi/2$

$$\Phi = \int_0^{\pi/2} \int_0^2 (r^2 + r^2 \sin\theta\cos\theta)\, \underbrace{r\,dr\,d\theta}_{dA} = 2\pi + 2$$

Ex Flux of $\vec{F}$ out of closed surface, $\vec{F} = \langle y, -x, z+1 \rangle$
$z = \sqrt{1 - x^2 - y^2}$



$$-\tfrac{1}{3}\pi = \Phi = \int_0^{2\pi} \int_0^1 (1 + (1-r^2)^{1/2})\, r\,dr\,d\theta \; - \text{area circle}$$