Surface Integrals for graphs

A surface S can be parametrized by two variables

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

For instance a sphere $\rho = 1$ has parametrization

$$\vec{r}(\phi,\theta) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

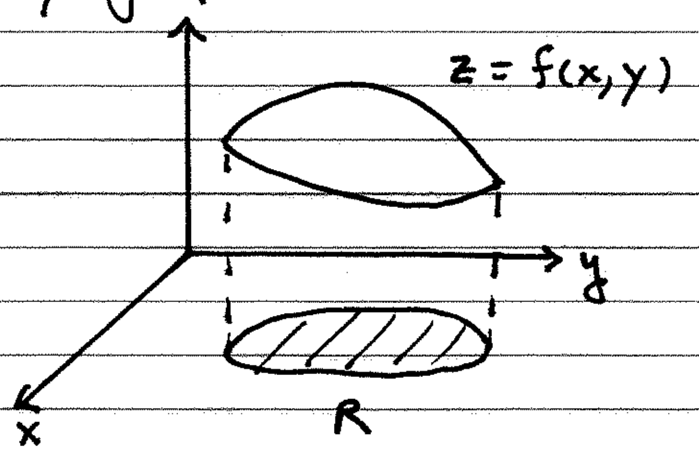
One can derive surface area elements

$$dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

for such a general class of surfaces.

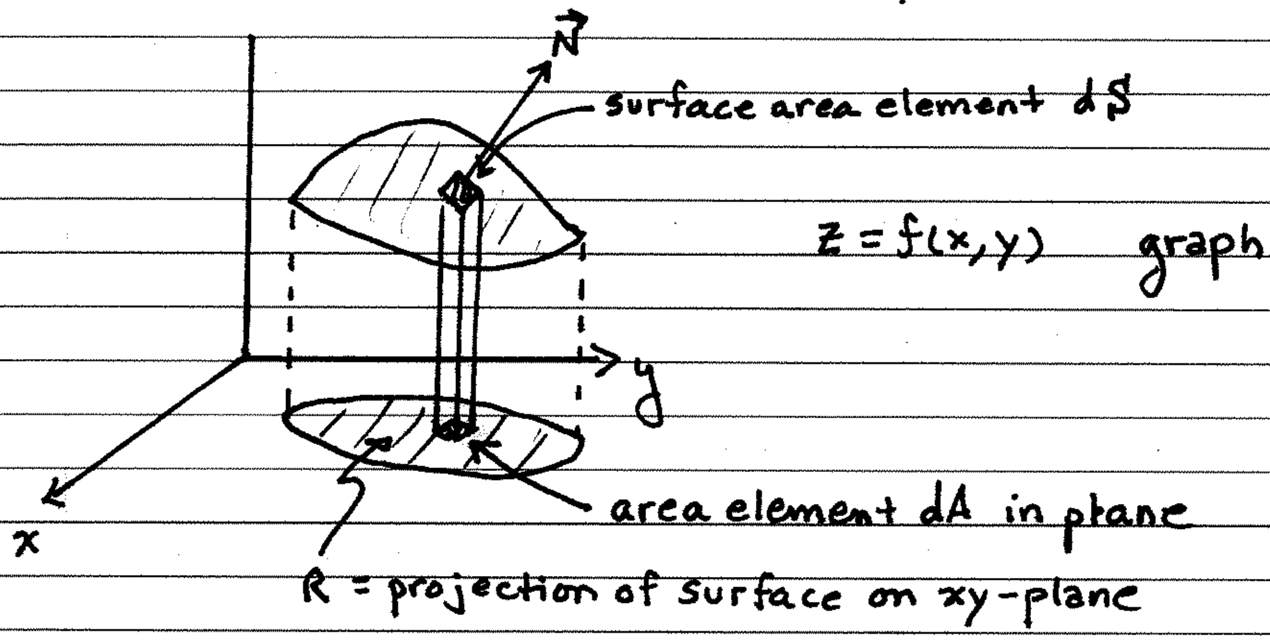
Such an approach works well for spheres, toroidal surfaces, etc.

We will restrict our attention to surfaces that can be described by graphs:

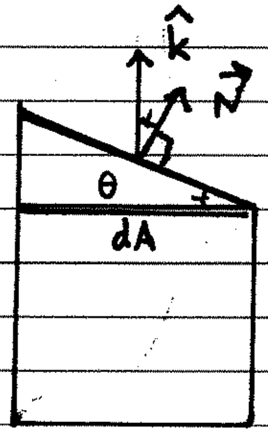


$(x,y) \in R$

Surface Area Elements for Graphs (16.4 formula 9)



Because of the slope of the surface one expects the surface element dS to be bigger than dA. From a side view



Normal to graph

$$\vec{N} = \langle -f_x, -f_y, 1 \rangle$$

and $dA = \cos\theta \, dS$.

But

$$\vec{N} \cdot \hat{k} = 1 = \|\vec{N}\| \cos\theta$$

Hence

$$\boxed{dS = \|\vec{N}\| \, dA = \sqrt{1 + f_x^2 + f_y^2} \, dA}$$

Thus the surface area of $\mathcal{S}$ is

$$(1) \quad S_A = \iint_{\mathcal{S}} d\mathcal{S} = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

where dA is either cartesian or polar $dA = r\,dr\,d\theta$.

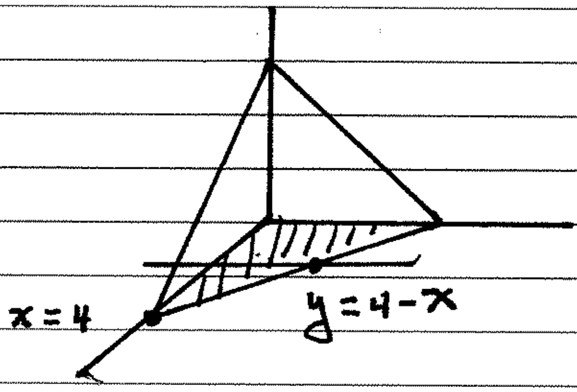
Surface integrals might involve "adding up" surface elements $d\mathcal{S}$ each having some density $F(x,y,z)$ say. Could be mass or charge denisty as in coulombs/m^2.

$$(2) \quad I = \iint_{\mathcal{S}} F(x,y,z)\, d\mathcal{S} = \iint_R F(x,y,f(x,y))\sqrt{1 + f_x^2 + f_y^2}\, dA$$

in this case I would be total net charge on surface $\mathcal{S}$.

Equations (1)-(2) are used to calculate surface integrals over some region R.

EX Surface area of plane $2x+2y+z=8$
in first octant $x,y,z>0$.



$$z = f(x,y) = 8 - 2x - 2y$$

$$\vec{N} = \langle -f_x, -f_y, 1\rangle = \langle 2, 2, 1\rangle$$

$$dS = \|\vec{N}\|\, dA$$

$$dS = 3\, dA$$

$$\boxed{\begin{array}{l} 0 < y < 4-x \\ 0 < x < 4 \end{array}}$$

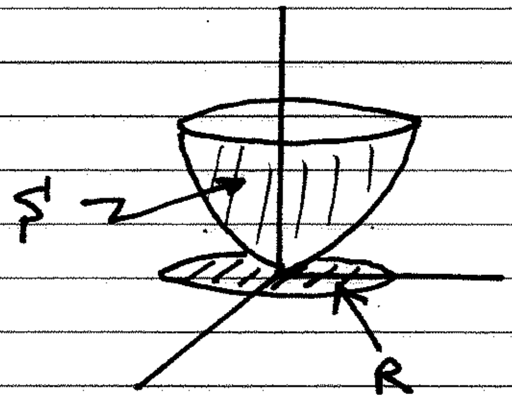
Surface area

$$S_A = \iint_S dS = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA$$

$$S_A = \int_0^4 \int_0^{4-x} 3\, dy\, dx$$ "set up"

$$S_A = 24$$

EX Surface area of the portion of $z = \frac{1}{2}(x^2+y^2)$ inside cylinder $x^2+y^2=8$



The regin in polar is

$0 < r < \sqrt{8}$

$0 < \theta < 2\pi$

Compute dS noting $\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle -2x, -2y, 1 \rangle$

$$\|\vec{N}\|^2 = 4x^2 + 4y^2 + 1$$

$$\|\vec{N}\| = \sqrt{1+4r^2}$$

$$dS = \sqrt{1+4r^2}\, dA \quad , \quad dA = r\,dr\,d\theta$$

Thus

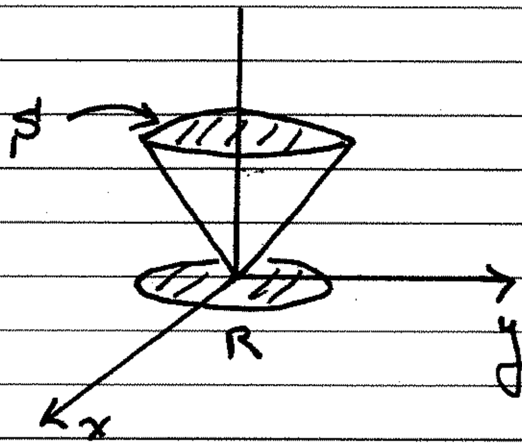
$$S_A = \iint_S dS = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA$$

$$S_A = \int_0^{2\pi}\int_0^{\sqrt{8}} \sqrt{1+4r^2} \cdot r\,dr\,d\theta \qquad \text{"set up"}$$

$(u = 1+4r^2)$

$$S_A = 2\pi\left(\frac{11\sqrt{33}}{4} - \frac{1}{12}\right)$$

Ex Portion of sphere $x^2+y^2+z^2=8$ inside the cone $z=\sqrt{x^2+y^2}$



$z=\sqrt{8-x^2-y^2}=f(x,y)$

Intersection of two surfaces

$z^2=8-r^2=r^2$

from which $r=2$

$$\boxed{\begin{array}{c} 0<r<2 \\ 0<\theta<2\pi \end{array}}$$

Calculate $\|\vec{N}\|$ where $\vec{N}=\langle -f_x, -f_y, 1\rangle$

$$f_x=-\frac{x}{\sqrt{8-r^2}} \qquad f_y=-\frac{y}{\sqrt{8-r^2}}$$

Hence

$$1+f_x^2+f_y^2=1+\frac{x^2}{8-r^2}+\frac{y^2}{8-r^2}=\frac{8}{8-r^2}$$

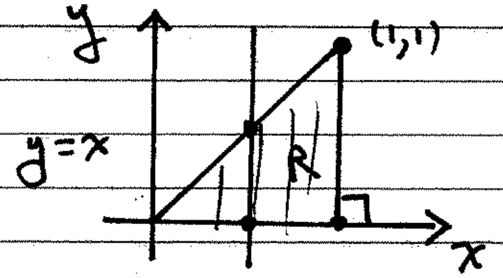
Then

$$S_A=\iint_S dS=\iint_S \sqrt{1+f_x^2+f_y^2}\,dA$$

$$S_A=\int_0^{2\pi}\int_0^2 \frac{\sqrt{8}}{\sqrt{8-r^2}}\,r\,dr\,d\theta=\cdots=16\pi$$

$(u=8-r^2)$

EX Portion of $z = 2x + y^2$ over region R which is triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$



$z = f(x,y) = 2x + y^2$

Region R given by

$0 < y < x$

$0 < x < 1$

Compute area element. First

$$\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle -2, -2y, 1 \rangle$$

$$dS = \|\vec{N}\|\, dA = \sqrt{5 + y^2}\, dA$$

Then

$$S_A = \iint_S dS = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$S_A = \int_0^1 \int_0^x \sqrt{5 + y^2}\, dy\, dx$$

too hard! interchange limits

$$S_A = \int_0^1 \int_y^1 \sqrt{5 + y^2}\, dx\, dy$$

Still too hard ... "setup" only.

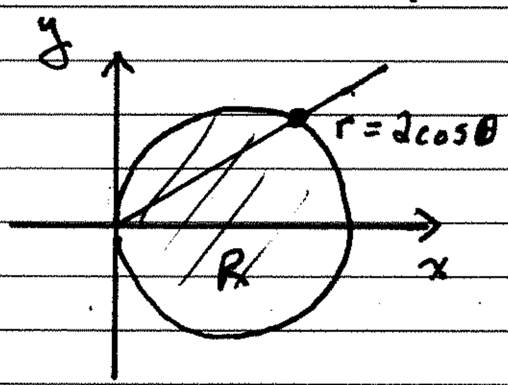
EX (Harder) Portion of cone inside cylinder $x^2+y^2=2x$, $z=\sqrt{x^2+y^2}$

Surface is hard to draw. Don't need to.

$$x^2+y^2=2x \iff \left.(x-1)^2+y^2=1\right\} \text{off center circle.}$$

In polar

$$\begin{aligned} x^2+y^2 &= 2x \\ r^2 &= 2r\cos\theta \\ r &= 2\cos\theta \end{aligned}$$



$$\boxed{\begin{gathered} 0 < r < 2\cos\theta \\ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \end{gathered}}$$

Compute area element

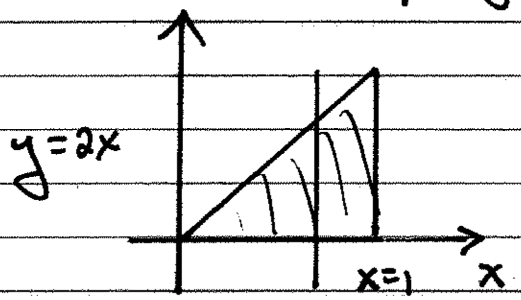
$$\vec{N} = \langle -f_x, -f_y, 1\rangle = \langle -\tfrac{x}{r}, -\tfrac{y}{r}, 1\rangle$$

$$\|\vec{N}\| = \sqrt{f_x^2+f_y^2+1} = \cdots = \sqrt{2}$$

Hence

$$S_A = \underbrace{\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}}_{R} \underbrace{\sqrt{2}}_{\|\vec{N}\|} \underbrace{r\,dr\,d\theta}_{dA} = \sqrt{2}\,\pi$$

EX Compute $\iint_S z^2\, dS$ where S is the portion of the plane $z = 4y$ over the triangle bounded by $y = 2x$, $x = 1$ and $y = 0$



$0 < y < 2x$
$0 < x < 1$

Compute area element dS where $f(x,y) = 4y$

$$\vec{N} = \langle -f_x, -f_y, 1\rangle = \langle 0, -4, 1\rangle$$

so $dS = \|\vec{N}\|\, dA = \sqrt{17}\, dA$

$$I = \iint_S z^2\, dS = \iint_R f(x,y)^2\, \|\vec{N}\|\, dA$$

or

$$I = \underbrace{\int_0^1 \int_0^{2x}}_{R} \underbrace{(4y)^2}_{z^2}\ \underbrace{\sqrt{17}}_{\|\vec{N}\|}\ \underbrace{dy\, dx}_{dA} = \frac{32\sqrt{17}}{3}$$

represents the mass of a plate whose density $F = z^2$ increases with height.