

Review Questions

(I) DOUBLE INTEGRALS - Cartesian and Polar Coordinates

1. Compute $\iint_R dA$ where R is the region bounded by $y = 3 - x$, and the x and y axes.
2. Evaluate $\iint_R (x - y) dA$ where R is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(2, 1)$.
3. Interchange the limits of integration and then evaluate:

$$a) \int_0^1 \int_y^1 e^{-x^2} dx dy \quad b) \int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x} dx dy \quad (1)$$

4. Use polar coordinates to evaluate $\iint_R (x + y) dA$ on the region in the first quadrant under $y = \sqrt{3}x$ and inside $x^2 + y^2 = 9$.
5. Set up the integral $\iint_R (x^2 + y^2) dA$ in polar coordinates where R is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.
6. Re-express the following iterated integral in polar coordinates:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} xy \, dy \, dx \quad (2)$$

7. Let R be the region in the first quadrant bounded by the y axis, the line $y = x$ and the circle $x^2 + y^2 = 4$. Draw R and then evaluate $\iint_R \sqrt{x^2 + y^2} dA$.
8. Evaluate $\iint_R x \, dA$ where R is the region inside the circle $x^2 + y^2 = 8$ with $y > x^2/2$.

(II) VOLUME INTEGRALS - Cartesian, Cylindrical(C), Spherical(S) Coordinates

1. Set up the cartesian and cylindrical integrals whose values are the volume of the region bounded by $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.
2. Set up the cartesian and cylindrical integrals whose values are the volume inside the cylinder $x^2 + y^2 = 4$, below the plane $z = x + y + 4$ and above $z = 0$.
3. Evaluate $\iiint_R (x^2 + y^2) dV$ on the cube $0 < x, y, z < 1$.
4. Evaluate $\iiint_R 3dV$ on the upper hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$.
5. Find the volume inside $x^2 + y^2 = 4$, above $z = 0$, below $z = x + 2$ and having $x > 0$.
6. Set up the iterated integral for $\iiint_R y dV$ where R is the portion of the cube $0 < x, y, z < 1$ above $y + z = 1$ and below $x + y + z = 2$. (Project onto yz -plane).
7. For each of the following regions, setup an integral for the volume

- a) (C) Above $z = (x^2 + y^2)^{1/4}$ and inside $x^2 + y^2 + z^2 = 1$.
- b) (C) Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$.
- c) (C) Above the xy -plane, under $z = 1 - x^2 - y^2$ and inside the wedge $-x \leq y \leq \sqrt{3}x$.
- d) (S) The interior of a sphere of radius 2 (center origin) less the portion with $z > \sqrt{x^2 + y^2}$

8. Find the mass of the upper hemisphere of radius 5 whose density is $f(x, y, z) = 10 - z$.
9. Let R be the region inside the cylinder $x^2 + y^2 = 1$, above the xy plane and below $z = y + 1$. The density of the solid is $F(x, y, z) = x^2 + y^2 + z^2$. Set up an iterated integral in cylindrical coordinates for the mass of the solid.
10. Use spherical coordinates to setup an iterated integral whose value is the mass of a solid R above $z = 0$, bounded by the unit sphere $x^2 + y^2 + z^2 = 1$ and the cones $z^2 = x^2 + y^2$, $z^2 = 3(x^2 + y^2)$. The density of the solid is $f(x, y, z) = z^2 + 1$ (Note: $z = \rho \cos \phi$ in spherical coordinates).

(III) LINE INTEGRALS

1. Let $\vec{F} = (x, x, z)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ on the curves

- (a) straight line from $(0,0,0)$ to $(1,1,1)$.
- (b) the curve parametrized by $\vec{r}(t) = (t, t^2, t^3), 0 \leq t \leq 1$.

2. Compute $\int_C x^2 dx$ on the unit circle (counterclockwise).
3. Compute $\int_C x dx + y dy$ on $y = x^2$, $x \in (-1, 1)$ traversed in a positive x direction.
4. Evaluate the line integral

$$\int_C y^2 dx \quad (3)$$

where C is the straight line from $(2, 1, 0)$ to $(1, 3, 7)$.

5. Compute the work done by $\vec{F} = (x^2 + xy, y - x^2y)$ along the path C parametrized by $\vec{r}(t) = (t, 1/t), 1 < t < 3$.

6. Evaluate

$$\int_C (y - x) dx + xy dy$$

where C is

- a) The straight line from $(1, 1)$ to $(2, 3)$.

b) Is the portion of the parabola $y = x^2$, $0 < x < 1$ directed in the positive x direction.

- X 7. Let C be given by $\vec{r}(t) = (t, 3t^2, 6t^3)$, $0 < t < 1$. If s is arclength, compute both of:

$$\int_C xyz^2 ds \quad , \quad \int_C xyz^2 dz$$

- X 8. A string of variable linear density $F(x, y, z) = x^2 + y^2$ (kg/m) has the shape of the helix $\vec{r}(t) = (2\cos t, 2\sin t, t)$, $0 < t < 2\pi$ (meters). Find its mass.

(IV) SURFACE INTEGRALS

1. Compute the surface area of the portion of the plane $x + 2y + z = 4$ inside the cylinder $x^2 + y^2 = 1$.
2. Compute the surface area of the paraboloid $z = 4 - x^2 - y^2$ above the $z = 1$ plane.
3. Set up a polar integral whose value is $\iint_S z dS$ where S is the portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$.
4. Evaluate $\iint_S z^2 dS$ on the upper portion of the sphere $x^2 + y^2 + z^2 = 1$.
5. Set up an iterated integral whose value is $\iint_S \vec{r} \cdot \vec{n} dS$, $\vec{r} = (x, y, z)$ and S being the plane $z = 1$ over the unit square $0 \leq x, y \leq 1$.
6. Find the surface area of the portion of the paraboloid $z = 2 - x^2 - y^2$ that is above the plane $z = 1$ and has $y \leq x$.
7. A surface S is the portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$. The surface has density $\rho(x, y, z) = x^2 z$ (kg/m²). Compute the mass.

(V) FLUX AND OTHER SURFACE INTEGRALS

1. Compute the flux of $\vec{F} = z\hat{i}$ through the upper unit hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ where the unit normal \hat{N} is oriented in the positive z direction.
2. Compute the flux of $\vec{F} = z\hat{k}$ through the portion of $x^2 + y^2 + z^2 = 4$ in the positive octant ($x, y, z > 0$) where the unit normal \hat{N} is oriented upward ($+z$ direction).
3. Compute the flux of $\vec{F} = (x, y, z)$ through the portion of the parabolic sheet $z = x^2$, $z \in (0, 4)$, $y \in (0, 1)$ with \hat{N} oriented upward ($+z$ direction).
4. Compute $\int_S \vec{F} \cdot \hat{N} dS$ where $\vec{F} = (z - x, 0, z + 1)$ and S is the portion of the plane $2x + y + z = 1$ in the positive octant and \hat{N} is oriented upwards.

(VI) CONSERVATIVE FIELDS

1. Use the curl to determine which of the following vector fields is conservative.

(i) $\vec{F}(x, y) = (xy - y, x - x^2)$

$$(ii) \vec{F}(x, y) = (y^2 - ye^{xy}, 2xy - xe^{xy})$$

$$(iii) \vec{F}(x, y, z) = (xy, xy, xz)$$

2 Find the associated potential function for each of the following vector fields

$$(i) \vec{F}(x, y) = (3x^2y + y, x^3 + x - 2y)$$

$$(ii) \vec{F}(x, y) = (ye^{xy} + \frac{1}{x+2y}, xe^{xy} + \frac{2}{x+2y})$$

$$(iii) \vec{F}(x, y, z) = (2xy, x^2 - z, 1 - y)$$

3 Evaluate the following line integrals

$$(i) \int_{(0,1)}^{(2,2)} (1+y)dx + (x+1)dy$$

$$(ii) \int_C \frac{1}{y} dx + \left(1 - \frac{x}{y^2}\right) dy \text{ where } C \text{ is parametrized by } \vec{r}(t) = (t+1, 1 + \sqrt{2t^2+1}), 0 < t < 2$$

$$(iii) \int_C dx + dy + 2zdz \text{ where } C \text{ is any simple curve from the origin to } (1,1,1).$$

(VII) DIVERGENCE THEOREM

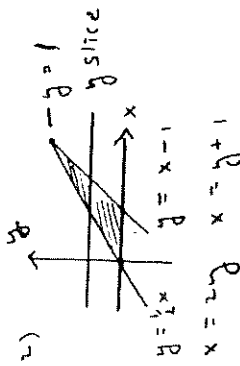
1 Let V be the solid with $\sqrt{x^2 + y^2} < z < 1$ and S its bounding surface. Compute the flux of $\vec{F} = (y^2x + z, y + 1, zx^2 - y)$ out of S .

2 Let V be the solid with $0 < z < 1 - y^2, 0 < x < 1$ and S its bounding surface. Compute the flux of $\vec{F} = (x, y, z)$ out of S .

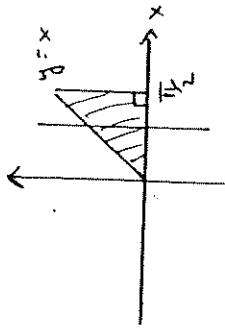
3 Use the divergence theorem to compute the flux of $\vec{F} = (y - z, yx^2, zy^2)$ through the portion of the paraboloid $z = x^2 + y^2, z < 1$ and the unit normal is oriented in the negative z direction (downward).

SECTION I

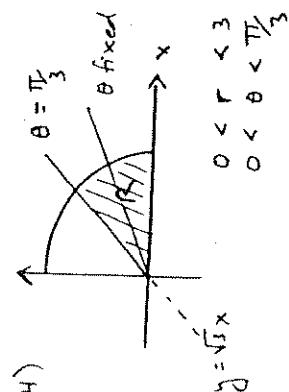
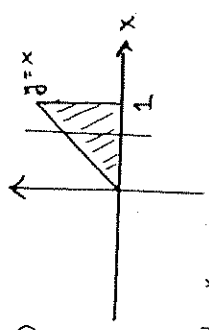
1) $\iint_R dA = \text{area of triangle} = \frac{3}{2}$



2) $\int_0^{y+1} \int_{2y}^{y+1} (x-y) dx dy = \frac{1}{3}$

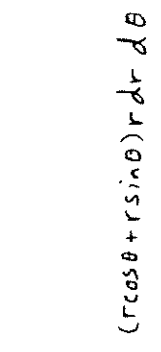


3) a) $\int_0^1 \int_0^x e^{-x^2} dy dx = \frac{1}{2}(1-e^{-1})$

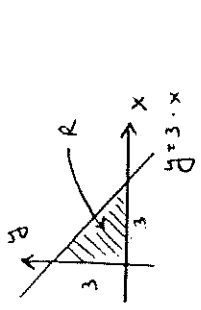


3) b) $\int_0^{\pi/3} \int_0^1 (r \cos \theta + r \sin \theta) r dr d\theta = \frac{2}{3}(\sqrt{3}+1)$

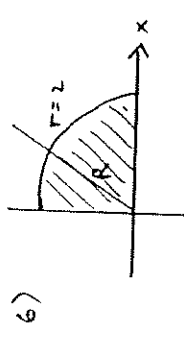
4) $\int_0^{\pi/2} \int_0^x \frac{\sin x}{x} dy dx = 1$



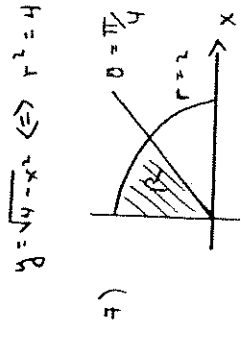
1.



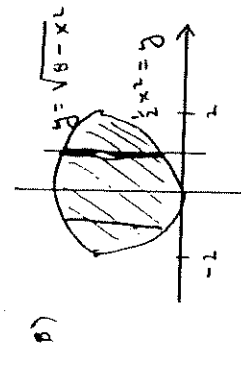
5) $\int_0^{\pi/4} \int_0^{\frac{1}{\cos \theta}} r^3 dr d\theta = \frac{1}{3}$ (HARD INTEGRAL)



6) $\int_0^{\pi/4} \int_0^2 r^2 \cos \theta \sin \theta dr d\theta = 2$



7) $\int_{\pi/4}^{\pi/2} \int_0^2 r^2 dr d\theta = \frac{4\pi}{3}$



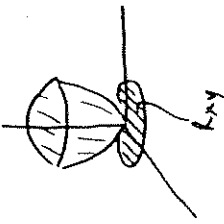
8) Intersection $x^2 + \frac{1}{4}x^4 = 8 \Rightarrow x=2$
 $\int_{-2}^2 \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \frac{1}{2}x^2 dy dx = 0$

2.

SECTION II

3.

1) $z = r^2 \quad z = \frac{1}{3}(4-r^2) \Rightarrow r=1$
 Intersection $r^2 = \frac{1}{3}(4-r^2) \Rightarrow r=1$
 $\int_0^{2\pi} \int_0^1 \int_0^{\frac{1}{3}(4-r^2)} r \, dz \, dr \, d\theta = \frac{2\pi}{3}$



Cartesian is $V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\frac{1}{3}(4-x^2-y^2)} dz \, dy \, dx$

2) $2\pi \int_0^2 \int_0^{4+\cos\theta+r\sin\theta} \int_0^{\sqrt{4-x^2-x^2+y^2+4}} r \, dz \, dr \, d\theta = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz \, dy \, dx$

3) $\int_0^1 \int_0^1 \int_0^1 (x^2+y^2) dz \, dy \, dx = \frac{2}{3}$

4) $\iiint_R 3 \, dV = 3 (\text{volume of hemisphere of radius 4}) = 2\pi(4)^3$

In other coordinates

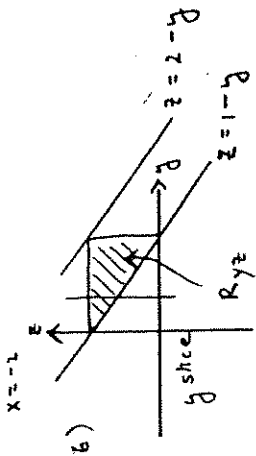
$3 \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^4 \int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

4.

5) $\int_{-\pi/2}^{\pi/2} \int_0^{r\cos\theta+2} \int_0^2 r \, dz \, dr \, d\theta = \frac{16}{3} + 4\pi$

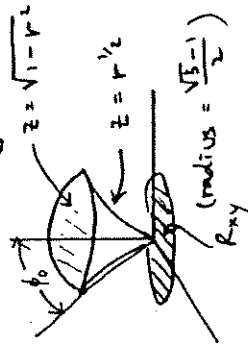
Project onto yz-plane

$x=0$
 $x = 2-y-z$
 On Ryz have $0 < 2-y-z < 1$



$\int_0^1 \int_{1-y}^{2-y-z} \int_0^2 y \, dz \, dy \, dx = \frac{5}{24}$

7 a) $z = (x^2+y^2)^{1/4} \Rightarrow z = r^{1/2}$ Intarseset



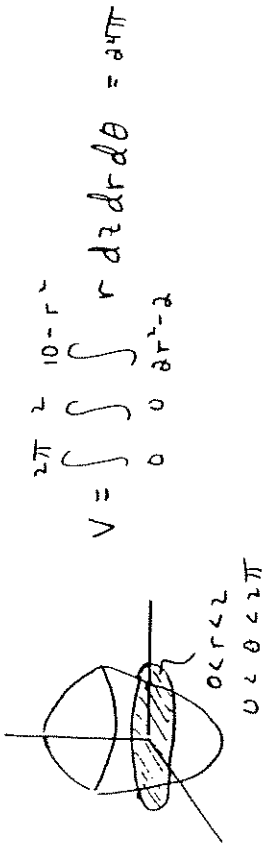
$r^2 + r - 1 = 0 \Rightarrow r = \frac{\sqrt{5}-1}{2}$

$V = \int_0^{2\pi} \int_0^{\frac{\sqrt{5}-1}{2}} \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$

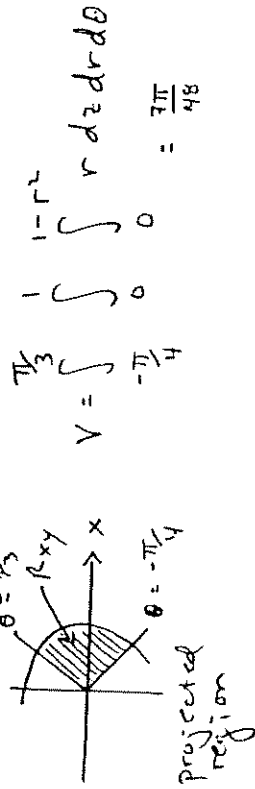
In spherical:

$V = \int_0^{2\pi} \int_0^{\phi_0} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$
 $\phi_0 = \arcsin\left(\frac{\sqrt{5}-1}{2}\right)$

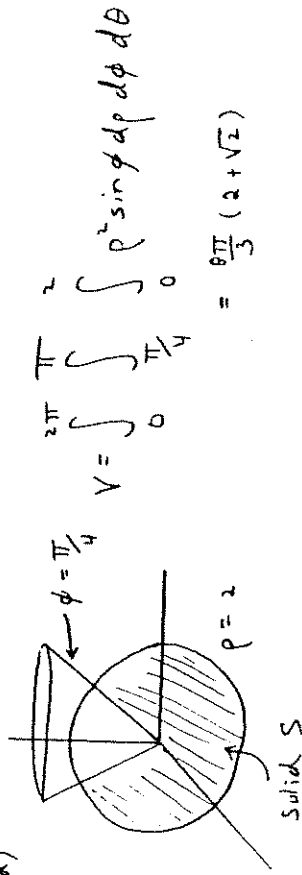
7b) $z = 10 - r^2$, $z = 2r^2 - 2$ intersect at $r = 2$ 5.



7c) Have $0 < z < 1 - r^2$ (Intersect $r = 1$)



7d)



8) $M = \iiint_V f dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 (10 - \rho \cos\phi) \rho^2 \sin\phi d\rho d\phi d\theta$

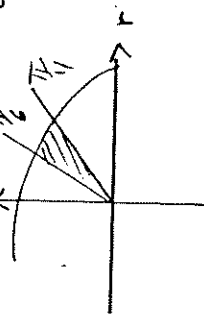
In cylindrical coordinates it would be

$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 (10 - z) r dz dr d\theta = \frac{8125\pi}{12}$ (MARR)

$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 (r^2 + z^2) r dz dr d\theta = \frac{13\pi}{12}$ (MARR)

9) Side view:

$z^2 = x^2 + y^2 \Rightarrow \phi = \frac{\pi}{4}$
 $z^2 = 3(x^2 + y^2) \Rightarrow \phi = \frac{\pi}{6}$



$M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\pi/6} (\rho \cos\phi + r) \rho^2 \sin\phi d\rho d\phi d\theta$

SECTION III

1a) $\vec{r}(t) = (t, t, t)$, $t \in (0, 1)$, $\vec{r}'(t) = (1, 1, 1)$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t, t, t) \cdot (1, 1, 1) dt = \frac{3}{2}$

1b) $\vec{r}(t) = (t, t^2, t^3)$, $t \in (0, 1)$, $\vec{r}'(t) = (1, 2t, 3t^2)$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t, t, t^3) \cdot (1, 2t, 3t^2) dt = \frac{5}{3}$

7.

$$a) \vec{r}(t) = (\cos t, \sin t), \quad t \in (0, 2\pi), \quad \vec{F} = (x^2, 0), \quad \vec{r}' = (-\sin t, \cos t)$$

$$\int_C x^2 dx = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\cos^2 t, 0) \cdot (-\sin t, \cos t) dt = 0$$

$$3) \vec{r}(t) = (t, t^2), \quad t \in (-1, 1), \quad \vec{F} = (x, y), \quad \vec{r}' = (1, 2t)$$

$$\int_C x dx + y dy = \int_{-1}^1 (t, t^2) \cdot (1, 2t) dt = 0$$

$$4) P = (2, 1, 0), \quad Q = (1, 3, 7), \quad \vec{r}(t) = \vec{OP} + t\vec{PQ}, \quad t \in (0, 1)$$

$$\vec{r}(t) = (2-t, 1+2t, 7t) \quad \vec{r}'(t) = (-1, 2, 7)$$

$$\text{Also, } \vec{F} = (y^2, 0, 0)$$

$$\int_C y^2 dx = \int_0^1 ((1+2t)^2, 0, 0) \cdot (-1, 2, 7) dt = -\frac{13}{3}$$

$$5) \vec{r}(t) = (t, \frac{1}{t}, t), \quad \vec{r}'(t) = (1, -\frac{1}{t^2}, 1), \quad t \in (1, 3)$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_1^3 (t^2+1, \frac{1}{t}-t) \cdot (1, -\frac{1}{t^2}, 1) dt = \frac{92}{3} + \ln 3$$

$$6a) \vec{r}(t) = (1+t, 1+2t), \quad \vec{r}' = (1, 2), \quad t \in (0, 1)$$

$$I = \int_0^1 (t, (1+t)(1+2t)) \cdot (1, 2) dt = \frac{11}{6}$$

8.

$$6b) \vec{r}(t) = (t, t^2), \quad t \in (0, 1), \quad \vec{r}' = (1, 2t)$$

$$I = \int_0^1 (t^2-t, t^3) \cdot (1, 2t) dt = \frac{7}{30}$$

$$7) \text{ Since } \vec{r}'(t) = (1, 6t, 18t^2),$$

$$ds = \|\vec{r}'(t)\| dt = \sqrt{(18t^2+1)^2} = \sqrt{18^2 t^4 + 6^2 t^2 + 1}$$

Thus

$$\int_C xy^2 ds = \int_0^1 108t^9 (18t^2+1) dt = \frac{864}{5}$$

For the second integral, $\vec{F} = (0, 0, xy^2)$, $\vec{r}' = (1, 6t, 18t^2)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (0, 0, 108t^9) \cdot (1, 6t, 18t^2) dt = 162$$

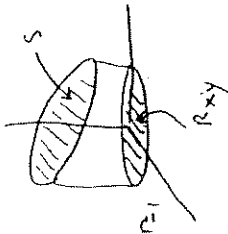
$$8) \vec{r}'(t) = (-2\sin t, 2\cos t, 1), \quad \|\vec{r}'(t)\| = \sqrt{5}$$

$$M = \int_C (x^2+y^2) ds = \int_0^{2\pi} (4) \sqrt{5} dt = 8\sqrt{5} \pi$$

SECTION III

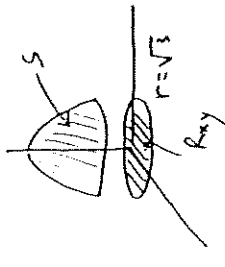
9.

1) $z = f(x, y) = 4 - x^2 - y^2$, $\|\vec{N}\| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{6}$



$$\iint_S ds = \sqrt{6} \iint_{R_{xy}} dA = \sqrt{6} \pi \cdot 1^2$$

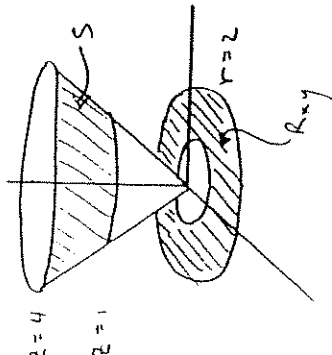
2) $z = f(x, y) = 4 - x^2 - y^2$, $\|\vec{N}\| = \sqrt{1 + 4r^2}$ in polar.



$$\iint_S ds = \int_0^{2\pi} \int_0^{\sqrt{3}} r \sqrt{1 + 4r^2} dr d\theta = \frac{\pi}{6} (13\sqrt{3} - 1)$$

where $dA = r dr d\theta$

3) $z = f(x, y) = \sqrt{x^2 + y^2}$, $z_x = \frac{x}{r}$, $z_y = \frac{y}{r}$, $\vec{N} = (-\frac{x}{r}, -\frac{y}{r}, 1)$
 then $\|\vec{N}\| = \sqrt{2}$.



$$\begin{aligned} \iint_S z ds &= \iint_{R_{xy}} \sqrt{2} \sqrt{x^2 + y^2} dA \\ &= \sqrt{2} \int_0^{2\pi} \int_0^2 r^2 dr d\theta \\ &= 42\pi\sqrt{2} \end{aligned}$$

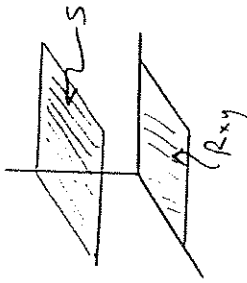
4) $z = f(x, y) = \sqrt{1 - x^2 - y^2}$, $\|\vec{N}\| = \frac{1}{\sqrt{1 - r^2}}$ in polar.

$$\iint_S z^2 ds = \iint_{x^2 + y^2 \leq 1} z^2 \cdot \|\vec{N}\| dA$$

$$I = \int_0^{2\pi} \int_0^1 (1 - r^2)(1 - r^2)^{-1/2} r dr d\theta$$

$$I = \int_0^{2\pi} \int_0^1 (1 - r^2)^{1/2} r dr d\theta = \frac{2\pi}{3}$$

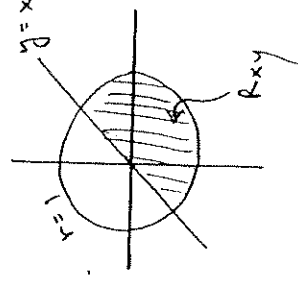
5) $z = f(x, y) = 1$, $\|\vec{N}\| = 1$, $ds = dA$



$$\vec{F} \cdot \vec{n} = x^2 + y^2 + z^2 \Big|_S = x^2 + y^2 + 1$$

$$\iint_S \vec{F} \cdot \vec{n} ds = \int_0^1 \int_0^1 (x^2 + y^2 + 1) dy dx = \frac{5}{3}$$

6) $z = 2 - x^2 - y^2$, $\|\vec{N}\| = \sqrt{1 + 4r^2}$



$$\begin{aligned} \iint_S ds &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r \sqrt{1 + 4r^2} dr d\theta \\ &= \frac{\pi}{12} (5\sqrt{5} - 1) \end{aligned}$$

10.

7) Similar to #3), $z = f(x, y) = \sqrt{x^2 + y^2}$, $\|\hat{N}\| = \sqrt{2}$

$$\iint_S x^2 z \, dS = \iint_{R_{xy}} \sqrt{2} x^2 \sqrt{x^2 + y^2} \, dA$$

$$\text{MASS } M = \int_0^{2\pi} \int_0^1 \sqrt{2} r^4 \cos^2 \theta \, dr \, d\theta = \frac{1023}{5} \pi \sqrt{2}$$

SECTION V

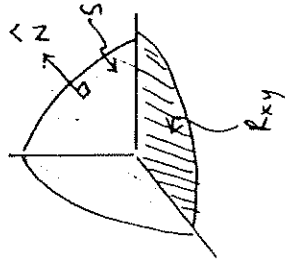
1) $\vec{F} = (z, 0, 0)$, $z = f(x, y) = \sqrt{1 - x^2 - y^2}$ and

$\vec{N} = (-f_x, -f_y, 1) = \left(\frac{x}{\sqrt{1-r^2}}, \frac{y}{\sqrt{1-r^2}}, 1\right)$ in mixed coord.

$$\iint_S \vec{F} \cdot \hat{N} \, dS = \iint_{R_{xy}} (\sqrt{1-r^2}, 0, 0) \cdot \left(\frac{x}{\sqrt{1-r^2}}, \frac{y}{\sqrt{1-r^2}}, 1\right) \, dA$$

$$\Phi = \int_0^{2\pi} \int_0^1 r^2 \cos \theta \, dr \, d\theta = 0$$

2) $\vec{F} = (0, 0, z)$, $z = f(x, y) = \sqrt{4 - x^2 - y^2}$, $\vec{N} = \left(\frac{x}{\sqrt{4-r^2}}, \frac{y}{\sqrt{4-r^2}}, 1\right)$



$$\Phi = \iint_{R_{xy}} \vec{F} \cdot \hat{N} \, dS = \iint_{R_{xy}} z \, dS$$

$$\Phi = \int_0^{2\pi} \int_0^2 r \sqrt{4 - r^2} \, dr \, d\theta = \frac{4\pi}{3}$$

3) $\vec{F} = (x, y, z)$, $z = f(x, y) = x^2$, $\vec{N} = (-2x, 0, 1)$

$$\Phi = \iint_{R_{xy}} -2x^2 + 0 + x^2 \, dA$$

$$\Phi = \int_{-2}^2 \int_0^1 (-x^2) \, dy \, dx = -\frac{16}{3}$$

$-2 < x < 2$
 $0 < y < 1$

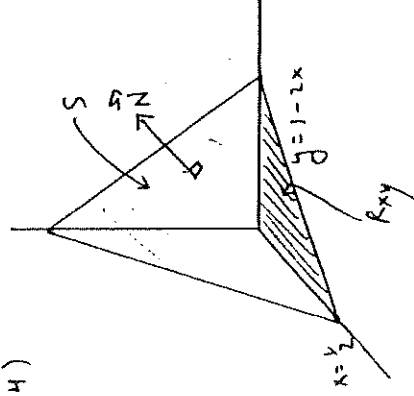
4)

$z = f(x, y) = 1 - 2x - y$

$\vec{N} = (2, 1, 1)$

$$\vec{F} \cdot \vec{N} = (z - x, 0, z + 1) \cdot (2, 1, 1) = 3z - 2x + 1$$

$$\vec{F} \cdot \vec{N} \Big|_S = 4 - 8x - 3y$$



$$\Phi = \int_0^{1/2} \int_0^{1-2x} (4 - 8x - 3y) \, dy \, dx = \frac{5}{12}$$

SECTION VII

1) i) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy - x^2 & z & 0 \end{vmatrix} = (z - 3x)\hat{k} \neq 0$ NOT CONS.

ii) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 - ye^{xy} & 2xy - xe^{xy} & 0 \end{vmatrix} = \vec{0}$ CONSERVATIVE

iii) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & \lambda y & xz \end{vmatrix} = (0, -z, y-x)$ NOT CONS.

2) i) $\phi_x = 3x^2y + \gamma \Rightarrow \phi(x, y) = x^3y + xy + \psi(y)$
 $\phi_y = x^3 + x - 2y = x^3 + x + \psi'(y) \Rightarrow \psi'(y) = -2y \Rightarrow \psi = -y^2$
 $\phi(x, y) = x^3y + xy - y^2$

ii) $\phi_x = ye^{xy} + \frac{1}{x+2y} \Rightarrow \phi(x, y) = e^{xy} + \log(x+2y) + \gamma(y)$
 $\phi_y = xe^{xy} + \frac{2}{x+2y} = xe^{xy} + \frac{2}{x+2y} + \psi'(y) \Rightarrow \psi = 0$
 $\phi(x, y) = e^{xy} + \log(x+2y)$

iii) $\phi_x = 2xy \Rightarrow \phi(x, y, z) = x^2y + f(y, z)$
 $\phi_y = x^2 - z = x^2 + \frac{\partial f}{\partial y} \Rightarrow f(y, z) = -yz + g(z)$
 $\phi_z = 1 - y = -y + g'(z) \Rightarrow g(z) = z$
 $\phi(x, y, z) = x^2y - yz + z$

3) Like question 2) must compute ϕ so that $\vec{F} = \nabla \phi$.

i) $\vec{F}(x, y) = (1+y, 1+x) \Rightarrow \phi(x, y) = x + xy + y$
 $P = (0, 1) \quad Q = (2, 2)$
 $\int_P^Q \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P) = 6 - 1 = 7$

ii) $\vec{F}(x, y) = (\frac{1}{y}, 1 - \frac{x}{y^2}) \Rightarrow \phi(x, y) = \frac{x}{y} + y$
 $P = F(0) = (1, 2) \quad Q = F(2) = (3, 4)$
 $\int_P^Q \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P) = 2\frac{1}{4}$

iii) $\vec{F}(x, y, z) = (1, 1, 2z) \Rightarrow \phi(x, y, z) = x + y + z^2$
 $P = (0, 0, 0) \quad Q = (1, 1, 1)$
 $\int_P^Q \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P) = 3$

VII DIVERGENCE THEOREM

1) $\nabla \cdot \vec{F} = x^2 + y^2 + 1$
 $\Phi = \oiint_S \vec{F} \cdot \hat{n} \, ds$
 $\Phi = \iiint_V (x^2 + y^2 + 1) \, dV$
 $\Phi = \int_0^{2\pi} \int_0^1 \int_0^1 (r^2 + 1) r \, dr \, d\theta \, dz$
 $\Phi = \frac{13}{30} \pi$

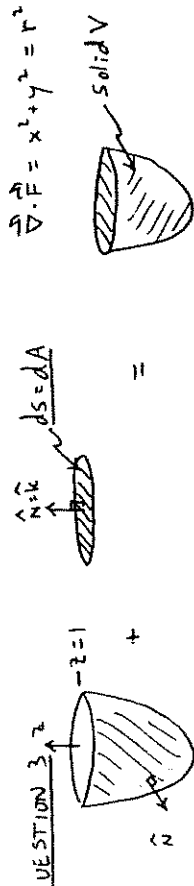
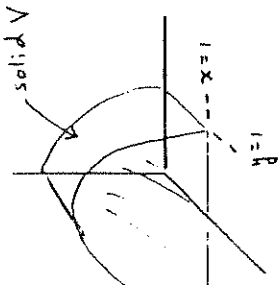
$z < 1$
 $0 < r < 1$
 $0 < \theta < 2\pi$

QUESTION 2

$$\nabla \cdot \vec{F} = 3$$

$$\Phi = \iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dV$$

$$\Phi = 3 \int_0^1 \int_{-1}^1 \int_{-1}^1 dz dy dx = 4$$



$$\nabla \cdot \vec{F} = x^2 + y^2 = r^2$$

$$\hat{n} \cdot \vec{k} ds = dA$$

$$\iint_{S_{BOT}} \vec{F} \cdot \hat{n} ds + \iint_{S_{TOP}} \vec{F} \cdot \hat{k} dA = \iiint_V \nabla \cdot \vec{F} dV$$

$$\iint_{S_{BOT}} \vec{F} \cdot \hat{n} ds + \int_0^{2\pi} \int_0^1 r^2 \sin^2 \theta dr d\theta = \int_0^{2\pi} \int_0^1 \int_0^1 r^2 dz dr d\theta$$

$$\iint_{S_{BOT}} \vec{F} \cdot \hat{n} ds + \frac{\pi}{4} = \frac{\pi}{6}$$

$$\iint_{S_{BOT}} \vec{F} \cdot \hat{n} ds = -\frac{\pi}{12}$$