1. (10) State which of the following sets $V$ are vector spaces. If it is not a vector space, state all the axioms (A1)-(A10) that are not satisfied.

a) Matrix addition and scalar multiplication and

$$ V = \{ u \in M_{22} : u = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \text{ for some } a, b \in \mathbb{R} \} $$

b) The set $V = \mathbb{R}^2$ where if $x, y \in V$ and $\alpha \in \mathbb{R}$ addition and scalar multiplication are defined by:

$$ x \oplus y = (x_1y_1, x_2y_2) $$

$$ \alpha \odot x = (\alpha x_1, \alpha x_2) $$

2. (20) Determine if each $W$ below is closed under addition and scalar multiplication. Then state if $W$ is/is not a subspace of the associated vector space $V$.

a) $W = \{ u \in M_{22} : u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ where } a + b + c + d = 0 \}, V = M_{22}$

b) $W = \{ f \in C(\mathbb{R}) : f(x) \leq 0 \}, V = C(\mathbb{R})$

c) $W = \{ f \in C(\mathbb{R}) : f(0) = 0 \}, V = C(\mathbb{R})$

d) $W = \{ A \in M_{nn} : A^T = -A \}, V = M_{nn}$

3. (15) For each of the following express $w$ as a linear combination of the vectors $v_k$ indicated. The vector space $V$ which $w, v_k$ are elements of are also indicated.

a) $V = \mathbb{R}^3, w = (7, 8, 9), v_1 = (2, 1, 4), v_2 = (1, -1, 3), v_3 = (3, 2, 5)$.

b) $V = P_2, w = 5x^2 + 13x + 3, v_1 = x^2 + 2x + 3, v_2 = -x^2 - 3x + 1$.

c) $V = M_{22}, w = \begin{bmatrix} -1 & -5 \\ 4 & 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

4. (10) In the following problems compute the coordinate $(w)_S$ of $w \in V$ relative the indicated bases. If, for example,

$$ S = \{ v_1, v_2, v_3 \} $$

where the vectors $v_k$ are defined then the coordinate $(w)_S$ is that vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ such that $w = c_1v_1 + c_2v_2 + c_3v_3$.

a) $V = \mathbb{R}^2, v_1 = (1, 2), v_2 = (-1, 1), S = \{ v_1, v_2 \}, w = (1, 1)$

b) $V = P_2, v_1 = x^2 + 1, v_2 = x + 1, v_3 = x - 1, S = \{ v_1, v_2, v_3 \}, w = x^2 + x + 1$.

5. (5) Let $S$ be the set of polynomials

$$ S = \{ x^3, x^2 + x, x^3 + x^2 + x, x^3 - x \} $$

and $W = \text{span}(S)$. Determine a basis $S'$ for $W$ which contains only elements of $S$. What is $\text{dim}(W)$?